Approximation Algorithms for CAs with Complement-Free Bidders

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Abstract

We exhibit two approximation algorithms for the allocation problem in combinatorial auctions. The running time of these algorithms is polynomial in the number of items m and in the number of players n, even though the "input size" is exponential in m. The first algorithm provides an $O(\log m)$ approximation for the case of complement-free bidders. The second algorithm provides an improved 2-approximation for the more restricted case of "XOS bidders", a class which strictly contains submodular bidders. We also prove lower bounds on the possible approximations achievable for these classes of bidders. These bounds are not tight and we leave the gaps as open problems.

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1 Introduction

In a combinatorial auction, a set M of items, $|M|=m$, is sold to n bidders. The combinatorial character of the auction comes from the fact that each bidder values *bundles* of items, rather than valuing items directly. I.e., the i'th bidder's value for each bundle is given by a valuation function v_i, where for each subset $S \subseteq M$, $v_i(S)$ denotes the value (maximum willingness to pay) of the bundle S for bidder i. We assume that for each bidder i, v_i is normalized (i.e. $v_i(\emptyset) = 0$), and monotone (for each $S \subseteq T \subseteq M$, $v_i(S) \leq v_i(T)$). The goal is to partition the items between the bidders in a way that maximizes the "social welfare" – the sum of players' values of the sets that they get. I.e. to find an allocation $S_1...S_m$, $S_i \cap S_j = \emptyset$ for $i \neq j$, that maximizes $\sum_i v_i(S_i)$. This problem is the common abstraction of many complex resource allocation problems both in computational settings and in economic settings and has received much attention both from the computational points of view and from the economic points of view – see the recent book [3].

The computational hardness of this problem is double: first, the "input" here is of exponential size – each v_i is described by 2^m real numbers, while we would like our algorithms to run in time polynomial in both m and n – i.e. in time that is poly-logarithmic in the input size. Second, even for valuations that are succinctly described, the optimization problem is computationally hard. Much work has thus been directed at identifying special cases that can be efficiently solved or approximated, as well as understanding the underlying computational limitations – see chapters 10 – 13 of [3].

Due to the exponential size of the input (that is, exponential in the parameters we care about), there are two approaches to formalize the computational model in which the allocation algorithms must work – specifically how the input valuations are specified and accessed. The first approach calls for fixing some "bidding language" in which the input valuations will be encoded, and requires the algorithms to run in polynomial time in the input length, under this encoding. This approach makes sense in cases for which a sufficiently natural bidding language exists. The second approach is a "concrete complexity" approach: the input valuations are given as "black boxes" and the type of queries that the algorithm may make to these input valuations is fixed. There are three types of query models that are commonly used:

1. **Value queries:** The query specifies a subset $S \subseteq M$ of items and receives the value $v_i(S)$ as the reply. This query is very natural from a "computer science" point of view, but in general is quite weak.

2. **Demand queries:** The query specifies a vector $p = (p_1...p_m)$ of "item prices", and the reply is the set that would be "demanded" by the queried bidder under these item prices. I.e., the subset S that maximizes $v_i(S) - \sum_{j \in S} p_j$. This query is natural from an economic point of view as it corresponds to "revealed preferences" of the bidders (i.e. what is directly observable from their behavior). It is known that these queries are strictly stronger than value queries (and in particular can simulate them in polynomial time) [2].

Valuation Class	Valuation	Demand	General Communication
General	$O(\frac{m}{\log m})$ [2] $\Omega(\frac{m}{\sqrt{\log m}})$ [2]	$O(m^{\frac{1}{2}})$ [2]	$\Omega(m^{\frac{1}{2}-\epsilon})$ [14]
CF	$O(m^{\frac{1}{2}})$	$O(\log m)$	≥ 2
XOS			≤ 2 $\geq \frac{4}{3}$
SM	≤ 2 [8] ≥ 1.02 (P$\neq$NP)		$\geq 1 + \frac{1}{2m}$ [14]
GS	1 [1]		

Figure 1: For each class of valuations, we specify the best approximation factor that is acheivable in polynomial time. models. Results without references are presented in this paper.

3. **General queries:** In this model we allow *any* kind of query to each valuation function (but the query is always to a single valuation function). This model captures the communication complexity (between the bidders) of the problem, and due to its strength is mostly interesting for lower bounds.

The computational complexity of the allocation problem with general valuations is almost completely understood: in polynomial time, the optimal allocation can be approximated to within a factor of $O(\sqrt{m})$ but not to a factor of $m^{1/2-\epsilon}$ for any $\epsilon > 0$. This is true both in the bidding langauge model for even single-minded bidders [9, 15]. The lower bound applies to general queries, where the upper bound requires demand queries [2], but value queries do not suffice.

In this paper we study the complexity of the allocation problem in the important special case where the input valuations are known not to have any complementarities. I.e., where all input valuations are sub-additive: $v(S \cup T) \leq v(S) + v(T)$ for all S, T [1]. In [8] a strict hierarchy of subclasses within this class of valuations is exhibited: $OXS \subset GS \subset SM \subset XOS \subset CF$. The classes CF and SM are easy to define: CF is the class of sub-additive (complement-free) valuations; SM is the set of submodular valuations, i.e., $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$ for all S, T. We will not define the class GS, "(gross) substitute" valuations here, but we will note that economists often assume valuations to be in this class as in some sense this corresponds to "convex economies". The classes OXS and XOS are defined syntactically as what can be defined by OR-of-XORs (resp. XOR-of-ORs) of singleton valuations. See [12] for definitions of the OR and XOR operations in this setting. For our purposes, the class XOS is the set of valuations v that can be represented by a $l \times m$ matrix p, where $v(S) = \max_{k=1...l} \sum_{j \in S} p_{kj}$.

[1] It is also possible to consider the "dual" class of substitute-free valuations ($v(S \cup T) \geq v(S) + v(T)$, for disjoint $S, T \subseteq M$). However, it turns out that the lower bound for general valuations [14], also stands for this class.

The allocation problem gets gradually harder within this hierarchy: a strongly polynomial time algorithm exists if the input valuations are given in the OXS language; a polynomial time algorithm using demand queries based on linear programming exists for the class GS [14]. For the class SM no polynomial time algorithm exists: NP-hardness for some simple bidding language is shown in [8] and an exponential communication lower bound is shown in [14]. However, [8] exhibit a polynomial-time 2-approximation algorithm that uses only value queries. No approximation algorithms (better than the $O(\sqrt{m})$-approximation for general valuations) were previously known for the higher levels in this hierarchy[2].

We provide new approximation algorithms for these two levels:

Theorem: There exists a polynomial time algorithm that finds a $O(\log m)$ approximation for valuations in the class CF using demand queries.

This algorithm is based on a careful randomized rounding of the linear programming formulation of the problem; a deterministic algorithm is obtained via derandomization.

We also provide an algorithm that uses the weaker value queries, and yields a worse approximation ratio. The main novelty in this algorithm is the fact that it is incentive compatible. I.e. the dominant strategy of all bidders is to always report their true valuations.

Theorem: There exists an *incentive compatible* polynomial time algorithm that finds a $O(\sqrt{m})$-approximation for valuations in the class CF using *value queries.*

For the more restricted class XOS we obtain an improved approximation ratio.

Theorem: There exists a polynomial time algorithm that finds a 2-approximation for valuations given in the XOS language.

The algorithm is greedy but very different from the algorithm of [8] designed for the more restricted SM class. We also provide a semantic characterization of the class XOS.

We prove lower bounds for approximation for CF and XOS. The class CF does not have a natural bidding language and so the lower bound is in the query model. The lower bound for the class XOS is actually two separate lower bounds: an NP-hardness result for the bidding language model, and a communication lower bound for the query model. No hardness result for approximation to within any constant factor for any of these classes was previously known.

Theorem: Exponential communication is required for approximating the optimal allocation among CF valuations to within any factor less than 2

Theorem: (1) It is NP-hard to approximate the optimal allocation among valuations given in the XOS language to within any constant factor less than $e/(e-1)$.

[2]This situation is similar to several other cases where submodular functions can be handled in sub-linear time, but nothing positive is known for more general functions. In particular this includes the celebrated algorithms for minimization of submodular functions [?, ?].

(2) Exponential communication is required for approximating the optimal allocation among XOS valuations to within any factor less than $4/3$.

Our results do not completely settle the complexity of approximate allocation among either submodular (SM) or sub-additive (CF) valuations (or XOS valuations). We suggest here the following "natural" conjectures:

Conjecture 1: Finding a better than $O(\log m)$-approximation of the optimal allocation among CF valuations requires an exponential amount of communication.

This would match our upper bound.

Conjecture 2: Finding a better than $O(\sqrt{m})$-approximation of the optimal allocation among CF valuations requires an exponential number of *value queries*.

This would match our upper bound, and highlight the gap between CF and SM (for which a 2-approximation using value queries exists), and also show that demand oracles are indeed required to achieve the $O(\log m)$ approximation.

Conjecture 3: There is a constant $c > 1$ such that any approximation of the optimal allocation for SM valuations to a factor better than c requires an exponential amount of communication.

This would strengthen the known lower bound of $1 + 1/(2m)$ of [14]. We are able to prove a result in this direction that is weaker in two aspects: the bound is only for value queries rather than general communication, and it is conditional upon $P \neq NP$.

Structure of the Paper

In section 2 we present the approximation algorithms for the class CF and the associated lower bound. Section 3 presents the algorithm for the class XOS and the associated lower bounds. Details of most proofs are postponed to the appendix.

2 Approximating Auctions with CF Valuations

Let us start with an imprecise description of the main idea of the algorithm. Randomized rounding of the LP-relaxation is a standard technique, and our algorithm uses it. However, when one attempts randomized rounding on packing problems such as combinatorial auctions, the results are not good: a randomized choice will very likely yield non-feasible solutions, unless the probabilities taken reduce the expected quality of solution by a large $O(\sqrt{m})$ factor. However, these non-feasible solutions are only a logarithmic factor away from feasibility. For general valuations this does not help, but this is the reason that the k-duplicate version of combinatorial auctions can be well approximated.

The main observation at the heart of our algorithm is that one may partition this logarithmically-non-feasible solution into a logarithmic size family of feasible solutions. For the case of complement-free valuations, the quality of one of these solutions may be bounded from below.

Let us now get more precise. We present here an algorithm for the allocation problem with n complement-free valuations. We assume that we have a demand oracle for each of the input valuations. In subsection 2.1 we first present a top-level description of the algorithm and then fill in the details of the steps. The proof of correctness appears in subsection 2.2.

In section 2.2 we describe an incentive compatible $O(\sqrt{m})$-approximation algorithm). In section 2.4 we present a lower bound – the lower bound is for a general communication model and thus applies to any algorithm that has any type of oracle access to the valuations.

2.1 The Algorithm

Input: The input is given as a set of n demand oracles for the n valuations v_i.

Output: An allocation T_1,...,T_n which is an $O(\log m)$ approximation to the optimal allocation.

The Algorithm: We first describe the basic steps of the algorithm and then provide the details regarding the implementation of each step.

1. Solve the linear relaxation of the problem: *Maximize:* $\Sigma_{i,S} x_{i,S} v_i(S)$ *Subject to:*
   - For each item j: $\Sigma_{i,S|j\in S} x_{i,S} \leq 1$
   - for each bidder i: $\Sigma_S x_{i,S} \leq 1$
   - for each i, S: $x_{i,S} \geq 0$
2. Use randomized rounding to find a "pre-allocation" $S_1, ..., S_n$ of pairs $< i, S_i >$ with the following properties, where $k = c \cdot \log(m)$, and $c > 0$ is a constant to be chosen later:
   - Each item j appears at most k times in $\{S_i\}_i$, with $j \in S_i$.
   - $\sum_i v_i(S_i) \geq \frac{1}{2} \cdot (\Sigma_{i,S} x_{i,S} v_i(S))$.
3. For each bidder i, partition S_i into a disjoint union $S_i = S_i 1 \cup ... \cup S_i^k$ such that for each $1 \leq i_1 < i_2 \leq n$ and $1 \leq r \leq k$, it holds that $S_{i_1}^r \cap S_{i_2}^r = \emptyset$.
4. Find the r that maximizes $\sum_i v_i(S_i^r)$, and for each i allocate $T_i = S_i^r$ to bidder i.

We now mention the details of each step:

1. **Solving the Linear Program:** Even though the linear program has exponentially many variables, it may still be solved in polynomial time. This is done by solving the dual linear program using the ellipsoid method. Using the ellipsoid method requires a "separation" oracle, and this may be directly implemented using the demand oracles of the bidders. Details appear in [14].

2. **Randomized Rounding:** For each bidder i we independently choose a set S_i by performing the following random experiment: each set S is chosen with probability $x_{i,S}$, and the empty set is chosen with probability $1 - \Sigma_S x_{i,S}$. If any of the required constraints is violated, then the this stage is repeated from scratch. This randomized step may be converted to be deterministic by derandomizing using the generator of [11] as explained in [16].

3. **Partitioning each S_i:** This is done as follows: for each $i = 1...n$ and each $r = 1..k$, we let $S_i^r = \{j \in S_i | j$ appears in exactly $r-1$ of the sets $S_1...S_{i-1}\}$.

4. **Choosing the best partition:** This step is straight forward.

Theorem 2.1 *If all input valuations are complement-free then the algorithm produces an allocation that is a $2k = O(\log m)$-approximation to the optimal one.*

2.2 Analysis

Proof: Let us keep track of the "quality" of solution implied by the intermediate steps.

1. The first step returns the optimal fractional solution $OPT^* = \Sigma_{i,S} x_{i,S} v_i(S)$, which is an upper bound to the value of the optimal allocation, OPT.

2. The detailed calculations needed to prove that this step indeed ends with a solution that satisfies all the required conditions are given in appendix A.1. At this point we will indicate the types of calculations used and what they yield. From the first inequality of the LP and using Chernoff bounds one can show that for every item j, the probability that it appears in more than k chosen sets is exponentially small in k. The expected value of $\sum_i v_i(S_i)$ at this stage is only slightly less than $\Sigma_{i,S} x_{i,S} v_i(S) = OPT^*$. It follows that with very high probability none of the required constraints are violated, and thus we have $\sum_i v_i(S) \geq \frac{1}{2} \cdot OPT^*$

3. The main point here is that indeed for every fixed r, the sets $\{S_i^r\}_i$ are pairwise disjoint and are thus a valid allocation. This follows directly from the construction, as every the duplicate instances of every item j are allocated to sets S_i^r with sequentially increasing r. Note that we always keep $r \leq k$ since each item appears in at most k sets in $\{S_i\}$.

4. The crucial use of complement-freeness comes here: since for each fixed i, $S_i = \bigcup_r S_i^r$, the fact that v_i is complement free implies that $\sum_r v_i(S_i^r) \geq v_i(S_i)$. By summing over all i we get that $\sum_r \sum_i v_i(S_i^r) = \sum_i \sum_r v_i(S_i^r) \geq \sum_i v_i(S_i) \geq \frac{1}{2} \cdot OPT^*$. It is now clear that by choosing the r that maximizes $\sum_i v_i(S_i^r)$ we get that $\sum_i v_i(S_i^r) \geq \frac{OPT^*}{2k}$. Thus the allocation $T_1 = S_1^r, ..., T_n = S_n^r$ is an $O(\log(m))$ approximation to the optimal allocation (and even to the optimal fractional allocation).

□

2.3 An Incentive-Compatible CF Auction With Value Queries

The only general technique known for making combinatorial auctions incentive compatible is the VCG mechanism (a definition of incentive compatibility and VCG can be found in appendix A.2). Unfortunately, using VCG requires solving the auction optimally - approximations of the optimal social welfare do not suffice [13, 9]. However, the amount of communication required for optimally solving combinatorial auctions is exponential.

In this section we present a way of overcoming this obstacle: limiting the set of possible allocations to a much simpler set. Optimal allocations within this set can be optimally found. Thus, VCG prices can be efficiently calculated, and incentive compatibility follows. Therefore, we are left with showing that the optimal solution within the restricted set of solutions always provides an approximation to the original problem.

The algorithm also answers another question: how well can the optimal social welfare be approximated using only valuation queries, given that all bidders' valuations are CF. The approximation ratio achieved in this section is $O(\sqrt{m})$.In contrast, for general valuations a lower bound of $\frac{m}{\log m}$ is known ([2]).
Let us now describe the algorithm:
Input: Each bidder i submits $m+1$ bids: $b_i(M)$, and $b_i(\{j\})$ for each $j \in M$.

Output: An allocation T_1,...,T_n which is an $O(\sqrt{m})$ approximation to the optimal allocation.

The Algorithm:
The algorithm chooses the best allocation out of the following two:

1. All items are allocated to the bidder that maximizes $b_i(M)$.
2. The best allocation in which each bidder gets at most one item.

Theorem 2.2 *If all the valuations are CF, the algorithm has the following properties:*

1. *Running time polynomial in n and m.*
2. *Provides an $O(\sqrt{m})$-approximation to the optimal allocation*
3. *Ensures incentive compatibility by using a pricing scheme computable in polynomial time.*

2.4 A Lower Bound

Theorem 2.3 *For every $\epsilon > 0$, any $(2-\epsilon)$-approximation algorithm for a combinatorial auction with bidders that have CF valuations, requires an exponential amount of communication.*

The proof of the theorem can be found in appendix A.4, and is based on showing a reduction from an auction presented in [14].

3 A 2-Approximation Algorithm for XOS Valuations

As mentioned before, it is known that $SM \subset XOS \subset CF$ (all inclusions are strict). While submodular auctions can be approximated to a factor of 2, no constant approximation factor was previously known for the higher levels of the hierarchy. The best known approximation ratio for CF auctions is $O(\log m)$, presented earlier. This section shows that a constant approximation ratio is achievable even for XOS auctions.

Unlike the semantically defined classes SM and CF, the class XOS was syntactically defined by [8]. This bidding language defines the largest valuations class known to be contained in CF. XOS is also very expressive and, in particular, contains the class of submodular valuations.

We offer two more reasons for the importance of XOS as a bidding language. First, there exist simple reductions from many interesting problems (e.g. MAX-3-SAT and MAX-k-COVER) to XOS auctions (see the lower bounds proofs for examples). Another reason is that we offer a natural semantic characterization to this syntactically defined class ("supporting prices").

The section begins by presenting the semantic characterization of XOS. We then go on to describing the algorithm itself and related lower bounds. Proofs not given in this section can be found in appendix B.

3.1 Supporting Prices

It turns out that XOS valuations have a unique property that makes XOS auctions approximable - existence of supporting prices for each bundle.

Definition 3.1 *A vector of prices $p_1 ... p_m$ $(\forall j \; p_j \geq 0)$ supports the bundle S in valuation v if:*

- $\forall j \notin S \; p_j = 0$
- $\sum_{j \in S} p_j = v(S)$.
- *For every bundle T,* $\sum_{j \in T} p_j \leq v(T)$.

The next proposition shows that existence of supporting prices for every bundle is a semantic characterization of the syntactic XOS class.

Proposition 3.2 *A valuation v is in the class XOS if and only if every bundle S has supporting prices.*

Our algorithm will be presented by using oracles for "supporting prices" queries as well as demand oracles.

Definition 3.3 *In a supporting prices query the question is a bundle S and the answer is a vector of supporting prices for it.*

In appendix B.1 we show that if the input is given in the form of an XOS expression, these two oracles can be simulated in time polynomial in the input size.

We do not know whether a supporting prices query can be simulated using only demand oracle queries if the input is not given in the XOS language. However, for the more restricted class of submodular valuations the following holds:

Proposition 3.4 *The supporting prices of submodular valuations can be calculated in polynomial time using only a valuation oracle.*

3.2 The Algorithm

This greedy algorithm chooses an arbitrary order of bidders and goes over them, one by one. Each bidder is asked for his demand bundle at given prices. The prices are updated after each step.

Input: n valuations v_i, for each of them we are given a demand oracle and a supporting prices oracle.

Output: An allocation S_1,...,S_n which is a 2 approximation to the optimal allocation.

The Algorithm:

1. Initialize $S_1 = ... = S_n = \emptyset$, and $p_1...p_m = 0$
2. For each bidder $i = 1...n$:
   (a) Let S_i be the demand of bidder i at prices $p_1...p_m$.
   (b) For all $i' < i$ take away from S'_i any items from S_i: $S_{i'} \leftarrow S_{i'} - S_i$.
   (c) Let $q_1...q_m$ be supporting prices for S_i in v_i.
   (d) For all $j \in S_i$, update $p_j = q_j$.

3.3 Analysis

Theorem 3.5 *The algorithm provides a 2 approximation to the optimal allocation.*

The full proof is in appendix B.4 and is obtained by putting together the two following lemmas:

Lemma 3.6 *The social welfare of the allocation generated by the algorithm is at least the sum of items' prices at the end of the algorithm (after the n'th stage).*

Lemma 3.7 *The social welfare of the optimal allocation is at most twice the sum of items' prices at the end of the algorithm.*

The example in B.5 shows that the algorithm does not achieve an approximation ratio better than 2. However, if all bidders have the same valuation the approximation guaranteed by the algorithm is improved to $\frac{e}{e-1}$. Theorem 3.9 shows that this ratio is the best possible, as no polynomial-time algorithm achieves a better ratio, unless $P = NP$.

Theorem 3.8 *If all bidders have the same XOS valuation, the algorithm provides an* $\frac{e}{e-1}$*-approximation ratio.*

A proof of this theorem can be found in appendix B.6.

3.4 Lower Bounds

There are two ways of getting the algorithm's input: one can assume that the input is given in XOS encoding and require the algorithm to run in time polynomial to the input size (rather than n and m). Another option, is to treat the valuations as "black boxes" and assume that we have supporting-prices and demand oracles.

For this reason we prove two lower bounds: The first is a lower bound in the NP-hardness model that applies when the input is in XOS encoding. The second lower bound is in communication and applies when the input is accessed via oracles.

Both proofs can be found in appendix B.

Theorem 3.9 *It is NP-hard to approximate the optimal allocation among valuations given in the XOS language to within any constant factor less than* $e/(e-1)$.

This theorem is proved by showing a reduction from MAX-k-COVER.

Theorem 3.10 *Exponential communication is required for approximating the optimal allocation among XOS valuations to within any factor less than* $4/3$

The theorem is proved by showing a reduction to the communication version of SET-COVER shown in [10]. The proof uses techniques from the proof of theorem 3.9 and from [4].

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A CF

A.1 Details of Stage 2

We will require the following version of the Chernoff bounds:

Theorem: (Chernoff Bound) Let $X_1,...X_n$ be independent Bernoulli trials such that for $1 \leq i \leq n$, $Pr[X_i = 1] = p_i$. Then for $X = X_1 + ... + X_n$, $\mu \geq p_1 + ...p_n$, and any $\delta \geq 2e - 1$ we have:

$$Pr[X > (1+\delta)\mu] < 2^{-\mu\delta}$$

We now prove that the last two conditions in stage 2 hold (with constant probability).

Lemma A.1 *The probability that there is an item that appears in more than k sets in $\{S_i\}$ is at most $\frac{1}{m^{c-2}}$.*

Proof: Fix an item j. Let $Z_{i,j}$ be the random variable that determines whether j appears in S_i. Obviously, $Z_{i,j}$ receives values in $\{0,1\}$. Because of the randomized rounding method we used, we have that the variables $\{Z_{i,j}\}_i$ are independent. We define $Z_j = \Sigma_i Z_{i,j}$ (i.e. Z_j is the number of times item j appears in $\{S_i\}$). By the linearity of expectation and the first condition of the LP formulation we have that $E[Z_j] \leq 1$. We can now use the Chernoff bound to get:

$$\Pr[\textit{item } j \textit{ appears in more than } k \textit{ bundles in } \{S_i\}] =$$

$$\Pr[Z_j > c \cdot \log(m)] < 2^{-(c-1)\cdot\log(m)} < \frac{1}{m^{c-1}}$$

By applying the union bound we get that the probability that any one of the items appears in more than k bundles in $\{S_i\}$ is smaller than $\frac{m}{m^{c-1}} = \frac{1}{m^{c-2}}$. □

Let A be the random variable that gets the value of $\Sigma_i v_i(S_i)$ after step 2. Let $B_{r,s}$ be the event that there is an item that appears at least r times in $\{S_i\}$, but no item appears more than s times in $\{S_i\}$. We will now bound from below the expectation of $E(A|B_{1,k})$ (i.e. the expectation given that each item was assigned to some bidder and no item appears in more than k sets in $\{S_i\}$).

Lemma A.2 $E(A|B_{1,k}) \geq \frac{1}{2} \cdot OPT^*$

Proof: Clearly,

$$OPT^* = E[A] = Pr[B_{1,k}]E(A|B_{1,k}) + \Sigma_{t=2}^{\infty} Pr[B_{(t-1)k+1,t\cdot k}]E(A|B_{(t-1)k+1,t\cdot k}) \quad [A.1]$$

on the other hand, for every $t \geq 2$, the following inequality holds:

$$\Pr[B_{(t-1)k+1,t\cdot k}]E(A|B_{(t-1)k+1,t\cdot k}) \leq \Pr[B_{(t-1)k+1,t\cdot k}](mtk\cdot OPT^*) \leq \frac{1}{m^{(t-1)(c-1)}}(mtk\cdot OPT^*)$$

To see that the first inequality holds, consider an extreme scenario in which each item appears exactly $t \cdot k$ times in $\{S_i\}$. Furthermore, assume that each item is given to a different bidder, and that each bidder values his item as OPT^*.

In the second inequality, we use an argument similar to the one in the previous lemma in order to bound the probability from above (Chernoff bound).
Observe, that for each $t \geq 2$, sufficiently large m, there is a choice of $c > 0$ such that:

$$\frac{1}{m^{(t-1)(c-1)}}(mtk \cdot OPT^*) \geq \frac{2}{m^{t(c-1)}}(m(t+1)k \cdot OPT^*)$$

therefore, there exists a geometric series, with a multiplier of $\frac{1}{2}$, that bounds the series of summands from above:

$$\Sigma_{t=2}^{\infty} \Pr[B_{(t-1)k+1,t\cdot k}]E(A|B_{(t-1)k+1,t\cdot k}) \leq \Sigma_{t=2}^{\infty}\frac{1}{m^{(t-1)(c-1)}}(mtk\cdot OPT^*) \leq \frac{2}{m^{c-2}}(2c(\log m)\cdot OPT^*)$$

We can now use equality A.1, to complete the proof of the lemma. □

We have shown that with good probability it is possible to create a a solution for which all the necessary conditions hold.

A.2 Definition of VCG

Theorem: (derived from [5]) Let b_i be a vector of bids submitted by the i'th bidder, such that $b_i(S)$ denotes the bid on bundle S (for every S). A combinatorial auction is incentive compatible if the payment p_i of the i'th bidder is defined to be:

$$p_i = \Sigma_{k\neq i} b_k(O_k) - \Sigma_{k\neq i} b_k(O_k^{-i})$$

where $O_1, ..., O_n$ is the optimal allocation (given b_1,...,b_n), and $O_1^{-i}, ..., O_n^{-i}$ is the optimal allocation in the same auction without the i'th bidder (given b_1,...,b_{i-1},b_{i+1},...,b_n).

A.3 Proof of Theorem 2.2

Proof:

1. Clearly, the first allocation can be calculated in polynomial time. Observe, that finding the second allocation is equivalent to finding the maximal weighted match in a bipartite graph: for each item j define a vertex a_j, and for each bidder i define a vertex b_i. Let the set of edges be $E = \cup_{i\in[n],j\in M}(a_j, b_i)$. Define the cost of each edge (a_j, b_i) to be $v_i(\{j\})$. Finding the maximal weighted match is equivalent to finding the best allocation in which each bidder gets at most one item. The maximal weighted match in bipartite graphs can be solved in polynomial time in m and n [**?**].

2. Let $OPT = \{T_1, ..., T_k, Q_1, ..., Q_l\}$ be the optimal allocation in the original auction, where for each $1 \leq i \leq k$, $|T_i| < \sqrt{m}$, and for each $1 \leq i \leq l$, $|Q_i| \geq \sqrt{m}$. Let $|OPT| = \Sigma_{i=1}^{l} v_i(Q_i) + \Sigma_{i=1}^{k} v_i(T_i)$.

   The first case we consider is when $\Sigma_{i=1}^{l} v_i(Q_i) \geq \Sigma_{i=1}^{k} v_i(T_i)$. Clearly, $\Sigma_{i=1}^{l} v_i(Q_i) \geq \frac{|OPT|}{2}$. Since $l \leq \sqrt{m}$ (otherwise, more than m items were allocated), for the bidder i that maximizes $v_i(O_i)$ it holds that $v_i(M) \geq v_i(Q_i) \geq \frac{|OPT|}{2\sqrt{m}}$. Thus, by assigning all items to bidder i we get the desired approximation ratio.

   We now consider the case where $\Sigma_{i=1}^{k} v_i(T_i) > \Sigma_{i=1}^{l} v_i(Q_i)$. Clearly, $\Sigma_{i=1}^{k} v_i(T_i) > \frac{|OPT|}{2}$. For each i, $1 \leq i \leq k$, let $c_i = \arg\max_{j \in T_i} v_i(\{j\})$. Notice, that $v_i(\{c_i\}) \geq \frac{v_i(T_i)}{|T_i|}$ (this is due to the CF property: $|T_i| \cdot v_i(\{c_i\}) \geq \Sigma_{j \in T_i} v_i(\{j\}) \geq v_i(T_i)$). Since for all i's $|T_i| < \sqrt{m}$, we have that: $\Sigma_{i=1}^{k} v_i(c_i) > \frac{\Sigma_i v_i(T_i)}{\sqrt{m}} \geq \frac{|OPT|}{2\sqrt{m}}$. By assigning c_i to bidder i we get an allocation in which every bidder gets at most one item with a social welfare of $\Sigma_{i=1}^{k} v_i(\{c_i\}) \geq \frac{|OPT|}{2\sqrt{m}}$. The second allocation, therefore, guarantees at least that social welfare.

   We conclude that the approximation the algorithm produces is at least $O(\sqrt{m})$.

3. We use VCG prices to obtain an incentive compatible combinatorial auction. The calculation of these prices requires solving an additional auction for each of the bidders. Note that these additional auctions are smaller in size (one less bidder). Thus, as was previously shown, the calculation of the prices can be done in polynomial time.

□

A.4 Proof of Theorem 2.4

Nisan and Segal [10] present an auction in which each bidder's valuation is restricted to values in $\{0, 1\}$. They show that distinguishing between the case that the optimal social welfare is 1, and the case that it is n, requires exponential communication.

Denote by v_i the valuation of the i'th bidder in the Nisan-Segal auction. Define new CF valuations in the following manner: $v_i'(S) = v_i(S) + 1$. One can easily verify that these new valuations are indeed CF.

Let us now consider a combinatorial auction with these valuations. We can see that distinguishing between the following cases requires exponential communication: the optimal social welfare is $n + 1$, and the optimal social welfare is $2n$. Hence, we have proved that for every $n \geq 2$ achieving $\frac{2n}{n+1}$-approximation requires exponential communication, and the theorem follows.

B XOS

The reader is referred to [8] for a more comprehensive discussion of the XOS class. The notation of [8] (in particular, for representing XOS valuations) will be used in

some of the proofs.

B.1 Simulating Oracles

Lemma B.1 *The following two queries can be answered for an XOS valuation given as an XOS expression in time polynomial in the input size:*

- *A demand query.*
- *A supporting prices query.*

Proof: Given an XOS valuation and a vector of prices we wish to simulate a demand oracle. First, let us note that one can easily simulate a demand oracle (in polynomial time) for an additive valuation (choose all the profitable items). Since the input is given as an XOS formula (in the form of a matrix) and each clause is an additive valuation, it is enough to simulate a demand oracle for each clause and choose the most profitable option. As there is only a polynomial number of such clauses the entire process requires polynomial time. The proof of proposition 3.2 contains a method of finding supporting prices to a given bundle of items. □

B.2 Proof of Proposition 3.2

Proof: First, we show that XOS valuation indeed has supporting prices. Given an XOS valuation and a set S we choose the clause that maximizes the value of S and assign the values of the items in the clause to the same items in S (other items are valued as 0). Obviously the sum of prices is indeed $v(S)$. The value of each $T \subseteq S$ is at least the sum of prices, since the clause sets a lower bound to the possible value of $v(T)$. In the other direction, given a valuation which has supporting prices, we will build an XOS valuation as follows: For each $S \subseteq M$ create a clause which consists of the supporting prices of S. Observe that the valuations are indeed equal. □

B.3 Proof of Proposition 3.4

Proof: To calculate the supporting prices for some group of $S \subseteq M$ items, first choose an arbitrary order of these items. Set the price p_j of the j'th item to be its marginal utility given the previous j-1 items (i.e. $v(1 \cup 2 \cup ... \cup j) - v(1 \cup 2 \cup ... \cup (j-1))$). Observe that $v(S) = \Sigma_{i \in S} p_i$. It's easy to see that for every $T \subseteq S$ $v(T) \geq \Sigma_{i \in T} p_i$, since by definition of SM the marginal utility does not increase when items are added. □

B.4 Proof of Theorem 3.5

Proof: For each $T \subseteq M$, we denote by $p^i(T)$ the total price of the items in subset T at the i'th stage of the algorithm. Let $\Delta^i = p^i(M) - p^{i-1}(M)$, i.e. the total difference in prices between stages $(i-1)$ and i (with $p0(M) = 0$). Let $A_1, ..., A_n$ be

the allocation generated by the algorithm. Let $O_1, ..., O_n$ be the optimal allocation. We will prove the $\Sigma_i v_i(O_i) \leq 2\Sigma_i v_i(A_i)$. To do so, we prove three simple lemmas:

Lemma B.2 *The social welfare of the allocation generated by the algorithm is at least the sum of items' prices at the end of the algorithm (after the n'th stage). I.e.* $p^n(M) \leq \Sigma_i v_i(A_i)$.

Proof: Consider a specific bidder i. Let T be the bundle assigned to that bidder by the algorithm in stage i. Obviously $A_i \subseteq T$. Since T corresponds to a specific clause in the XOS formula, we have that $p^i(A_i) \leq v_i(A_i)$ (due to the supporting prices property). However, since $p^i(A_i) = p^n(A_i)$ (the items in A_i were not reassigned after the i'th stage, and so their prices were not altered), we have that $p^n(A_i) \leq v_i(A_i)$, and so $p^n(M) = \Sigma_{i=1}^n p^n(A_i) \leq \Sigma_{i=1}^n v_i(A_i)$. □

Lemma B.3 *The prices assigned to the items throughout the algorithm are non-decreasing.*

Proof: By contradiction. Let $v_i(S)$ maximize the demand given a prices vector p, i.e. $v_i(S) = max_{i' \in [n], T \subseteq [m]} v_{i'}(T)$. Let q be a supporting prices vector for S. Now, assume there is an item $j \in S$ for which $q_j < p_j$. The supporting prices property ensures that $\Sigma_{t \in (S-\{j\})} q_t \leq v_i(S - \{j\})$ and $\Sigma_{r \in S} q_r = v_i(S)$. Hence:

$$v_i(S) - \Sigma_{r \in S} p_r = \Sigma_{r \in S} q_r - \Sigma_{r \in S} p_r = (q_j - p_j) + (\Sigma_{t \in (S-\{j\})} q_t - \Sigma_{t \in (S-\{j\})} p_t) <$$

$$(\Sigma_{t \in (S-\{j\})} q_t - \Sigma_{t \in (S-\{j\})} p_t) < v_i(S - \{j\}) - \Sigma_{t \in (S-\{j\})} p_t$$

and this is a contradiction to the way $v_i(S)$ was chosen. □

Lemma B.4 *The social welfare of the optimal allocation is at most twice the sum of items' prices at the end of the algorithm. I.e.* $\Sigma_i v_i(O_i) \leq 2p^n(M)$.

Proof: Since for each i, $1 \leq i \leq n$, $\Delta^i = \max_{T \subseteq M}(v_i(T) - p^{i-1}(T))$, we have:

$$v_i(O_i) - p^{i-1}(O_i) \leq \Delta^i.$$

since the prices do not decrease throughout the algorithm, the following inequality holds:

$$v_i(O_i) - p^n(O_i) \leq \Delta^i.$$

by summing up on both sides of the equation we get:

$$\Sigma_{i=1}^n v_i(O_i) - \Sigma_i p^n(O_i) \leq \Sigma_i \Delta^i$$

$$\Sigma_i v_i(O_i) - p^n(M) \leq p^n(M)$$

$$\Sigma_i v_i(O_i) \leq 2p^n(M)$$

□

Putting the lemmas together we have that

$$\Sigma_i v_i(O_i) \leq 2p^n(M) \leq 2\Sigma_i v_i(A_i)$$

□

B.5 Example of Approximation Ratio of 2

Consider a combinatorial auction with two goods, x_1 and x_2, and two bidders. The first bidder's valuation is $v_1(\{x_1\}) = v_1(\{x_2\}) = v_1(\{x_1 \cup x_2\}) = 1$. The valuation of the second bidder is $v_2(\{x_1\}) = 0,\ v_2(\{x_2\}) = v_2(\{x_1 \cup x_2\}) = 1$. Clearly, a social welfare of 2 can be achieved by allocating x_1 to the first bidder, and x_2 to the second bidder. However, the first bidder might wish to get x_2 at the first stage, and the optimal social welfare achieved is only 1. Hence, the approximation ratio achieved by the algorithm is only 2.

B.6 Proof of Theorem 3.8

Proof: The proof of the theorem relies on the proof of [7] which studies the Budgeted Maximum Coverage Problem. We note that this proof, and the techniques presented in that paper can be used to prove the results of [7] in a more general setting. However, this is beyond the scope of this paper. The notation throughout this proof is the same as in the proof of theorem 3.5. To prove the theorem we will first prove two lemmas:

Lemma B.5 *After iteration i of the algorithm we have that*

$$\frac{1}{n}(\Sigma_i v_i(O_i) - p^{i-1}(M)) \leq p^i(M) - p^{i-1}(M)\ (= \Delta^i).$$

Proof: By definition $\Delta^i = max_{T \subseteq M}(v_i(T) - p^{i-1}(T))$. Hence, for each j, $1 \leq j \leq n$

$$v_i(O_j) - p^{i-1}(O_j) \leq \Delta^i.$$

Since all bidders have the same valuation the following inequality holds:

$$v_j(O_j) - p^{i-1}(O_j) \leq \Delta^i.$$

By summing up on j (on both sides of the equation) we get:

$$\Sigma_i v_i(O_i) - p^{i-1}(M) \leq n\Delta^i.$$

and the lemma follows. □

Lemma B.6 *After the i'th iteration of the algorithm the following holds:*

$$(1 - (1 - \frac{1}{n})^i)(\Sigma_i v_i(O_i)) \leq p^i(M)$$

Proof: We prove this lemma by induction on i. For i=1, $p^1(M) = v_1(M) \geq \frac{1}{n}\Sigma_i v_i(O_i)$. If we assume correctness for $(i-1)$ then:

$$p^i(M) = p^{i-1}(M) + \Delta^i$$

by the previous lemma and the induction step,

$$p^i(M) \geq p^{i-1}(M) + \frac{1}{n}(\Sigma_i v_i(O_i) - p^{i-1}(M)) = \frac{1}{n}\Sigma_i v_i(O_i) + (1 - \frac{1}{n})p^{i-1}(M) \geq$$

$$\frac{1}{n}\Sigma_i v_i(O_i) + (1 - \frac{1}{n})(1 - (1 - \frac{1}{n})^{i-1})(\Sigma_i v_i(O_i)) = (1 - (1 - \frac{1}{n})^i)(\Sigma_i v_i(O_i))$$

□

Using the previous lemma we get:

$$\frac{e-1}{e}\Sigma_i v_i(O_i) \leq (1 - (1 - \frac{1}{n})^n)(\Sigma_i v_i(O_i)) \leq p^n(M) \leq \Sigma_i v_i(A_i)$$

the right inequality was proved in theorem 3.5. □

B.7 Proof of Theorem 3.9

Proof: We will show a polynomial-time reduction from MAX-k-COVER. MAX-k-Cover is defined as follows: Given m items, and a collection of subsets of these items, the objective is to maximize the number of items which can be covered by k subsets. It is known that it is NP-hard to approximate this problem within a better factor than $\frac{e}{e-1}$ [4]. This problem can easily be converted into a combinatorial auction with XOS valuations: given an instance of MAX-k-COVER, we create an auction with k bidders and m goods. Each bidder has the same XOS valuation in which there is a clause for each subset in the original problem, and the value of every item in the clause is 1. The items in each clause are connected with ORs and the clauses are connected with XORs. Clearly, every choice of k subsets in the original problem corresponds to an allocation in the combinatorial auction with the same value. All we need to do is to assign all items in set i to bidder i (and avoid assigning one item to more then one bidder). In the other direction, every allocation corresponds to a choice of k sets in the original problem with at least the social welfare value: We choose k subsets, so that subset i contains the items in the clause maximizing bidder i's gain. This way we are guaranteed that the number of items covered is no less than the social welfare. The theorem follows. □

B.8 Proof of Theorem 3.10

Proof: In [10] it is shown that any algorithm achieving a $(\frac{1}{2}\log m - \epsilon)$ approximation for the SET-COVER problem (as defined there) requires exponential communication. In [4] (proposition 12), it is shown how to deduce a lower bound for MAX-k-COVER by a reduction from SET-COVER. An analogous method produces a communication lower bound of $\frac{4}{3}$ to the communication version of MAX-k-COVER. In this version of the problem we have k players, each holding a collection of subsets of $1..m$. We are interested in maximizing the number of items covered by k subsets when subset i can only be chosen from player i's subsets. An argument similar to the

one in theorem 3.9 shows that there is an approximation preserving reduction from this version of MAX-k-COVER to an auction where the bidders have different XOS valuations. Hence, any algorithm with approximation ratio better than $\frac{4}{3}$ requires exponential communication. We note that assuming the set-cover communication lower bound can be improved to $\ln(m)$ (as is the case with the computational lower bound), the threshold specified in this theorem can be improved to $\frac{e}{e-1}$. □

C A Constant Lower Bound for Approximating Submodular Auctions

Theorem C.1 *Using valuation queries only, it is NP-hard to approximate the optimal allocation among submodular to within any constant factor less than* 1.02.

Proof: We will prove the lower bound by reducing from MAX-CUT. MAX-CUT is the problem of finding a partition of vertices in a graph G, such that the cut (the number of edges with one vertex in either side of the partition) is maximized. Given a graph $G = (V, E)$, define a two bidder auction as follows: the goods will be the vertices, and the two bidders will have an identical valuation functions: $v(S) = |\cup_{(i,j)\in E | i\in S}(i,j)|$. Observe that these valuations are indeed submodular, and that for every S, $v(S)$ can be calculated in polynomial time. Each allocation defines a cut, and each cut defines an allocation. The total social welfare of an allocation is $|E|$, plus the number of edges which have been counted twice (once for each bidder). The edges which have been counted twice, are the cut defined by the allocation. Denote by ALG the number of edges that have been counted twice in the combinatorial auction, and by OPT the number of edges in the maximum cut. Recalling that the maximum cut consists of at least half of the edges, we have that

$$\frac{|E| + ALG}{|E| + OPT} \leq \frac{2OPT + ALG}{3OPT} = \frac{2}{3} + \frac{ALG}{3OPT}$$

[6] sets a lower bound of $\frac{17}{16} - \epsilon$ for approximating MAX-CUT. We can now conclude that it is NP-hard to approximate combinatorial auctions with submodular valuations within a factor of $\frac{51}{50} - \epsilon$. □