Stochastic Dual Coordinate Ascent Methods for Regularized Loss Minimization

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"Optimization for Machine Learning",
NIPS Workshop, 2012

Regularized Loss Minimization

$$\min_w P(w) := \left[\frac{1}{n}\sum_{i=1}^{n}\phi_i(w^\top x_i) + \frac{\lambda}{2}\|w\|^2\right].$$

Regularized Loss Minimization

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Examples:

	$\phi_i(z)$	Lipschitz	smooth
SVM	$\max\{0, 1 - y_i z\}$	✓	✗
Logistic regression	$\log(1 + \exp(-y_i z))$	✓	✓
Abs-loss regression	$\|z - y_i\|$	✓	✗
Square-loss regression	$(z - y_i)^2$	✗	✓

Dual Coordinate Ascent (DCA)

Primal problem:

$$\min_w P(w) := \left[\frac{1}{n} \sum_{i=1}^{n} \phi_i(w^\top x_i) + \frac{\lambda}{2} \|w\|^2 \right]$$

Dual problem:

$$\max_{\alpha \in \mathbb{R}^n} D(\alpha) := \left[\frac{1}{n} \sum_{i=1}^{n} -\phi_i^*(-\alpha_i) - \frac{\lambda}{2} \left\| \frac{1}{\lambda n} \sum_{i=1}^{n} \alpha_i x_i \right\|^2 \right]$$

- DCA: At each iteration, optimize $D(\alpha)$ w.r.t. a single coordinate, while the rest of the coordinates are kept in tact.
- Stochastic Dual Coordinate Ascent (SDCA): Choose the updated coordinate uniformly at random

SDCA vs. SGD — update rule

Stochastic Gradient Descent (SGD) update rule:

$$w^{(t+1)} = \left(1 - \tfrac{1}{t}\right) w^{(t)} - \frac{\phi_i'(w^{(t)\top} x_i)}{\lambda t} x_i$$

SDCA update rule:

$$1.\quad \alpha^{(t+1)} = \operatorname*{argmax}_{\Delta \in \mathbb{R}} D(\alpha^{(t)} + \Delta\, e_i)$$

$$2.\quad w^{(t+1)} = w^{(t)} + \frac{\Delta}{\lambda n} x_i$$

- Rather similar update rules.
- SDCA has several advantages:
    - Stopping criterion
    - No need to tune learning rate

SDCA vs. SGD — update rule — Example

SVM with the hinge loss: $\phi_i(w) = \max\{0, 1 - y_i w^\top x_i\}$

SGD update rule:

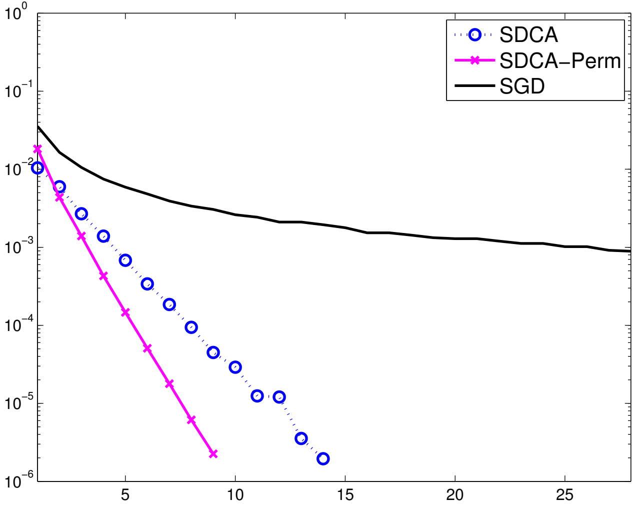
$$w^{(t+1)} = \left(1 - \tfrac{1}{t}\right) w^{(t)} - \frac{\mathbf{1}[y_i\, x_i^\top w^{(t)} < 1]}{\lambda\, t}\, x_i$$

SDCA update rule:

1. $\Delta = y_i \max\left(0, \min\left(1, \dfrac{1 - y_i\, x_i^\top w^{(t-1)}}{\|x_i\|_2^2/(\lambda n)} + y_i\, \alpha_i^{(t-1)}\right)\right) - \alpha_i^{(t-1)}$
1. $\alpha^{(t+1)} = \alpha^{(t)} + \Delta\, e_i$
2. $w^{(t+1)} = w^{(t)} + \dfrac{\Delta}{\lambda\, n}\, x_i$

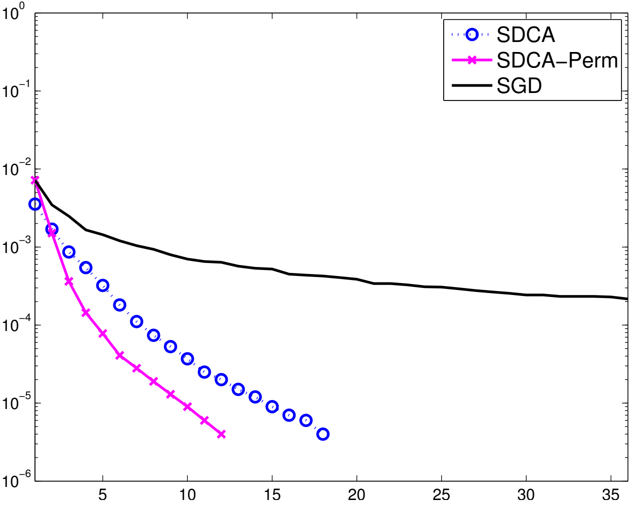
SDCA vs. SGD — experimental observations

- On CCAT dataset, $\lambda = 10^{-6}$, smoothed loss



SDCA vs. SGD — experimental observations

- On CCAT dataset, $\lambda = 10^{-5}$, hinge-loss



SDCA vs. SGD — Current analysis is unsatisfactory

How many iterations are required to guarantee $P(w^{(t)}) \leq P(w^*) + \epsilon$?

- For SGD: $\tilde{O}\left(\frac{1}{\lambda \epsilon}\right)$
- For SDCA:
  - Hsieh et al. (ICML 2008), following Luo and Tseng (1992): $O\left(\frac{1}{\nu} \log(1/\epsilon)\right)$, but, ν can be arbitrarily small
  - S and Tewari (2009), Nesterov (2010):
    - $O(n/\epsilon)$ for general n-dimensional coordinate ascent
    - Can apply it to the dual problem
    - Resulting rate is slower than SGD
    - And, the analysis does not hold for logistic regression (it requires smooth dual)
  - Analysis is for dual sub-optimality

Dual vs. Primal sub-optimality

- Take data which is linearly separable using a vector w_0
- Set $\lambda = 2\epsilon/\|w_0\|^2$ and use the hinge-loss
- $P(w^*) \le P(w_0) = \epsilon$
- $D(0) = 0 \;\Rightarrow\; D(\alpha^*) - D(0) = P(w^*) - D(0) \le \epsilon$
- But, $w(0) = 0$ so $P(w(0)) - P(w^*) = 1 - P(w^*) \ge 1 - \epsilon$
- Conclusion: In the "interesting" regime, ϵ-sub-optimality on the dual can be meaningless w.r.t. the primal !

Our results

- For $(1/\gamma)$-smooth loss:

$$\tilde{O}\left(\left(n+\frac{1}{\lambda}\right)\log\frac{1}{\epsilon}\right)$$

- For L-Lipschitz loss:

$$\tilde{O}\left(n+\frac{1}{\lambda\,\epsilon}\right)$$

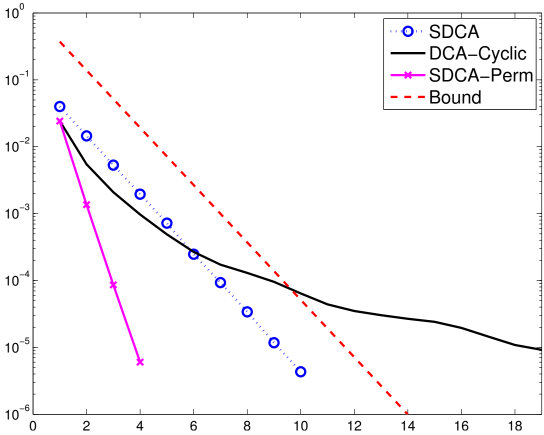
- For "almost smooth" loss functions (e.g. the hinge-loss):

$$\tilde{O}\left(n+\frac{1}{\lambda\,\epsilon^{1/(1+\nu)}}\right)$$

where $\nu > 0$ is a data dependent quantity

SDCA vs. DCA — Randomization is crucial

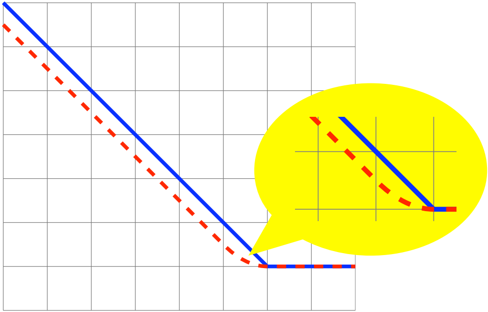
- On CCAT dataset, $\lambda = 10^{-4}$, smoothed hinge-loss



- In particular, the bound of Luo and Tseng holds for cyclic order, hence must be inferior to our bound

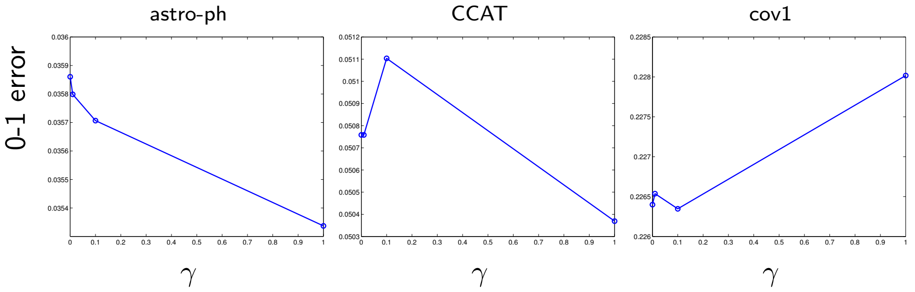
Smoothing the hinge-loss

$$\phi(x) = \begin{cases} 0 & x > 1 \\ 1 - x - \gamma/2 & x < 1 - \gamma \\ \frac{1}{2\gamma}(1-x)^2 & \text{o.w.} \end{cases}$$



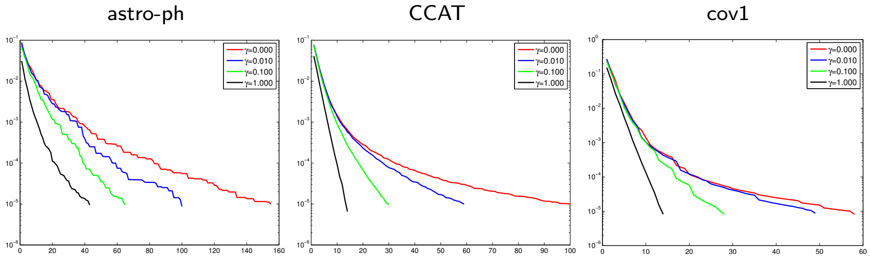
Smoothing the hinge-loss

- Mild effect on 0-1 error



Smoothing the hinge-loss

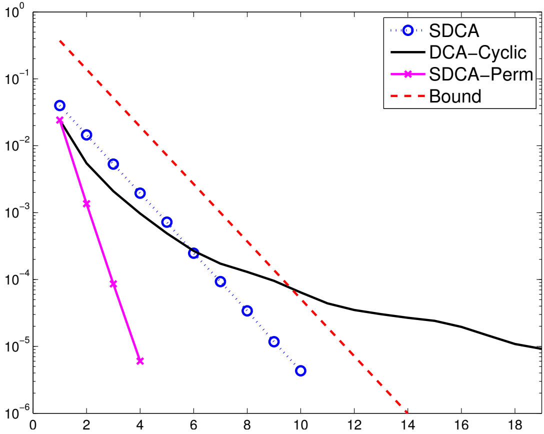
- Improves training time



- Duality gap as a function of runtime for different smoothing parameters

SDCA vs. DCA — Randomization is crucial

- On CCAT dataset, $\lambda = 10^{-4}$, smoothed hinge-loss



- In particular, the bound of Luo and Tseng holds for cyclic order, hence must be inferior to our bound

Additional related work

- Collins et al (2008): For smooth loss, similar bound to ours (for smooth loss) but for a more complicated algorithm (Exponentiated Gradient on dual)
- Lacoste-Julien, Jaggi, Schmidt, Pletscher (preprint on Arxiv):
  - Study Frank-Wolfe algorithm for the dual of structured prediction problems.
  - Boils down to SDCA for the case of binary hinge-loss.
  - Same bound as our bound for the Lipschitz case
- Le Roux, Schmidt, Bach (NIPS 2012): A variant of SGD for smooth loss and finite sample. Also obtain $\log(1/\epsilon)$.

Extensions

- Slightly better rates for SDCA with SGD initialization
- "Proximal Stochastic Dual Coordinate Ascent" (in preparation, a preliminary version is on Arxiv)
  - Solve:

$$\min_w P(w) := \left[\frac{1}{n} \sum_{i=1}^{n} \phi_i(X_i^\top w) + \lambda g(w) \right]$$

  - For example: $g(w) = \frac{1}{2}\|w\|_2^2 + \frac{\sigma}{\lambda}\|w\|_1$ leads to thresholding operator
  - For example: $\phi_i : \mathbb{R}^k \to \mathbb{R}$, $\phi_i(v) = \max_{y'} \delta(y_i, y') + v_{y_i} - v_{y'}$ is useful for structured output prediction
  - Approximate dual maximization (can obtain closed form approximate solutions while still maintaining same guarantees on the convergence rate)

Proof Idea

- Main lemma: for any t and $s \in [0,1]$,

$$\mathbb{E}[D(\alpha^{(t)}) - D(\alpha^{(t-1)})] \geq \frac{s}{n}\, \mathbb{E}[P(w^{(t-1)}) - D(\alpha^{(t-1)})] - \left(\frac{s}{n}\right)^2 \frac{G^{(t)}}{2\lambda}$$

- $G^{(t)} = O(1)$ for Lipschitz losses
- With appropriate s, $G^{(t)} \leq 0$ for smooth losses

Proof Idea

- Main lemma: for any t and $s \in [0,1]$,

$$\mathbb{E}[D(\alpha^{(t)}) - D(\alpha^{(t-1)})] \geq \frac{s}{n}\, \mathbb{E}[P(w^{(t-1)}) - D(\alpha^{(t-1)})] - \left(\frac{s}{n}\right)^2 \frac{G^{(t)}}{2\lambda}$$

- Bounding dual sub-optimality:
  Since $P(w^{(t-1)}) \geq D(\alpha^*)$, the above lemma yields a convergence rate for the dual sub-optimality
- Bounding duality gap: Summing the inequality for iterations $T_0+1,\ldots,T$ and choosing a random $t \in \{T_0+1,\ldots,T\}$ yields,

$$\mathbb{E}\left[(P(w^{(t-1)}) - D(\alpha^{(t-1)}))\right] \leq \frac{n}{s(T-T_0)}\, \mathbb{E}[D(\alpha^{(T)}) - D(\alpha^{(T_0)})] + \frac{s\,G}{2\lambda n}$$

Summary

- SDCA works very well in practice
- So far, theoretical guarantees were unsatisfactory
- Our analysis shows that SDCA is an excellent choice in many scenarios