On the tradeoff between computational complexity and sample complexity in learning

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- Prior knowledge: $h^\star \in \mathcal{H}$

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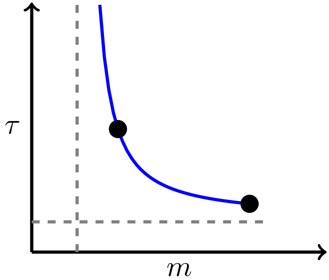
  - Data laden — $\mathrm{err}(\infty,\tau)$

Main Conjecture

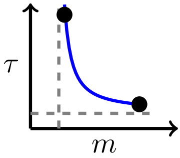
Main Question

How much time, τ, is needed to achieve error $\leq \epsilon$ as a function of sample size, m?



Warmup example

- $\mathcal{X} = \{0, 1\}^d$



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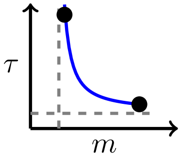

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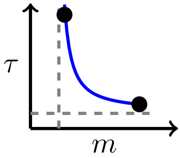

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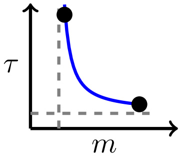

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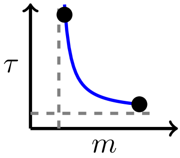
  - $|\mathcal{H}| \leq 3^{3d}$ therefore sample complexity is order d/ϵ



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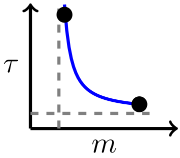

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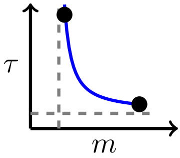
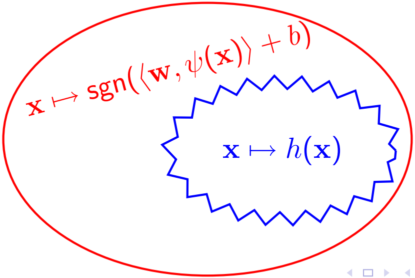
  - $|\mathcal{H}| \leq 3^{3d}$ therefore sample complexity is order d/ϵ
  - Kearns & Vazirani: If RP≠NP, it is not possible to efficiently find $h \in \mathcal{H}$ s.t. $\text{err}(h) \leq \epsilon$
- Claim: if $m \geq d^3/\epsilon$ it is possible to find a predictor with error $\leq \epsilon$ in polynomial time



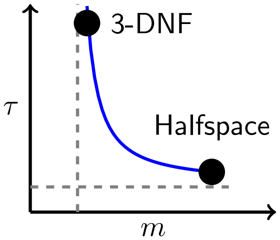
How more data reduces time?

- Observation: $T_1 \vee T_2 \vee T_3 = \wedge_{u \in T_1, v \in T_2, w \in T_3} (u \vee v \vee w)$
- Define: $\psi : \mathcal{X} \to \{0,1\}^{2(2d)^3}$ s.t. for each triplet of literals u, v, w there are two variables indicating if $u \vee v \vee w$ is true or false
- Observation: Exists Halfspace s.t. $h^\star(\mathbf{x}) = \text{sgn}(\langle \mathbf{w}, \psi(\mathbf{x}) \rangle + b)$
- Therefore, can solve ERM w.r.t. Halfspaces (linear programming)
- VC dimension of Halfspaces is the dimension
- Sample complexity is order d^3/ϵ



Trading samples for runtime

Algorithm	samples	runtime
3-DNF	$\frac{d}{\epsilon}$	2^d
Halfspace	$\frac{d^3}{\epsilon}$	$\text{poly}(d)$



But,

- The lower bound on the computational complexity is only for *proper* learning — there's no lower bound on the computational complexity of improper learning with d/ϵ examples
- The lower bound on the sample complexity of Halfspaces is in the general case — here we have a specific structure

The interesting questions:

- Is the curve really true ? Can one construct 'correct' lower bounds ?
- If the curve is true, one should be able to construct more algorithms on the curve. How?

Second example: Online Ads Placement

For $t = 1, 2, \ldots, m$

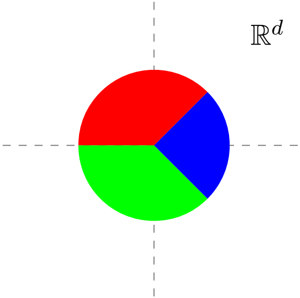
- Learner receives side information $\mathbf{x}_t \in \mathbb{R}^d$
- Learner predicts $\hat{y}_t \in [k]$
- Learner pay cost $\mathbf{1}[\hat{y}_t \neq h^\star(\mathbf{x}_t)]$
- "Bandit setting" — learner does not know $h^\star(\mathbf{x}_t)$

Goal: Minimize error rate:

$$\text{err} = \frac{1}{m} \sum_{t=1}^{m} \mathbf{1}[\hat{y}_t \neq h^\star(\mathbf{x}_t)] \ .$$

Linear Hypotheses

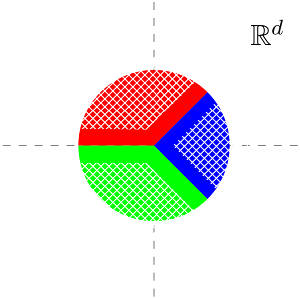
$$\mathcal{H} = \{\mathbf{x} \mapsto \underset{r}{\text{argmax}}\,(W\,\mathbf{x})_r \;\;:\;\; W \in \mathbb{R}^{k,d},\; \|W\|_F \le 1\}$$



Large margin assumption

Assumption: Data is separable with margin μ:

$$\forall t,\ \forall r \neq y_t,\ (W\mathbf{x}_t)_{y_t} - (W\mathbf{x}_t)_r \geq \mu$$



First approach – Halving

Halving for Bandit Multiclass categorization

Initialize: $V_1 = \mathcal{H}$

For $t = 1, 2, \ldots$

- Receive $\mathbf{x}_t$
- For all $r \in [k]$ let $V_t(r) = \{h \in V_t : h(\mathbf{x}_t) = r\}$
- Predict $\hat{y}_t \in \arg\max_r |V_t(r)|$
- If $\mathbf{1}[\hat{y}_t \neq y_t]$ set $V_{t+1} = V_t \setminus V_t(\hat{y}_t)$

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Analysis:

- Whenever we err $|V_{t+1}| \leq \left(1 - \frac{1}{k}\right) |V_t| \leq \exp(-1/k)\, |V_t|$
- Therefore: $\text{err} \leq \frac{k \log(|\mathcal{H}|)}{m}$
- Equivalently, sample complexity is $\frac{k \log(|\mathcal{H}|)}{\epsilon}$

Using Halving

- Step 1: Dimensionality reduction to $d' = O(\frac{\ln(m+k)}{\mu^2})$
- Step 2: Discretize $\mathcal{H}$ to $(1/\mu)^{kd'}$ hypotheses
- Apply Halving on the resulting finite set of hypotheses

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Analysis:

- Sample complexity is order of $\frac{k^2/\mu^2}{\epsilon}$
- But runtime grows like $(1/\mu)^{kd'} = (m+k)^{\tilde{O}(k/\mu^2)}$

How can we improve runtime?

- Halving is not efficient because it does not utilize the structure of $\mathcal{H}$
- In the full information case: Halving can be made efficient because each version space V_t can be made convex !
- The Perceptron is a related approach which utilizes convexity and works in the full information case
- Next approach: Lets try to rely on the Perceptron

The Mutliclass Perceptron

For $t = 1, 2, \ldots, m$

- Receive $\mathbf{x}_t \in \mathbb{R}^d$
- Predict $\hat{y}_t \;=\; \arg\max_r (W^t\,\mathbf{x}_t)_r$
- Receive $y_t = h^\star(\mathbf{x}_t)$
- If $\hat{y}_t \neq y_t$ update: $W^{t+1} = W^t + U^t$

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$$U^t = \begin{bmatrix} 0 & \ldots & 0 \\ & \vdots & \\ 0 & \ldots & 0 \\ \ldots & \mathbf{x}_t & \ldots \\ 0 & \ldots & 0 \\ & \vdots & \\ 0 & \ldots & 0 \\ \ldots & -\mathbf{x}_t & \ldots \\ 0 & \ldots & 0 \\ & \vdots & \\ 0 & \ldots & 0 \end{bmatrix}$$

Row y_t

Row $\hat{y}_t$

Problem: In the bandit case, we're blind to value of y_t

The Banditron (Kakade, S, Tewari 08)

- Explore: From time to time, instead of predicting $\hat{y}_t$ guess some $\tilde{y}_t$
- Suppose we get the feedback 'correct', i.e. $\tilde{y}_t = y_t$
- Then, we have full information for Perceptron's update: $(\mathbf{x}_t, \hat{y}_t, \tilde{y}_t = y_t)$

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- Exploration-Exploitation Tradeoff:
  - When exploring we may have $\tilde{y}_t = y_t \neq \hat{y}_t$ and can learn from this
  - When exploring we may have $\tilde{y}_t \neq y_t = \hat{y}_t$ and then we had the right answer in our hands but didn't exploit it

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For $t = 1, 2, \ldots, m$

- Receive $\mathbf{x}_t \in \mathbb{R}^d$
- Set $\hat{y}_t = \arg\max_r (W^t \mathbf{x}_t)_r$
- Define: $P(r) = (1 - \gamma)\mathbf{1}[r = \hat{y}_t] + \frac{\gamma}{k}$
- Randomly sample $\tilde{y}_t$ according to P
- Predict $\tilde{y}_t$
- Receive feedback $\mathbf{1}[\tilde{y}_t = y_t]$
- Update: $W^{t+1} = W^t + \tilde{U}^t$

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- Receive feedback $\mathbf{1}[\tilde{y}_t = y_t]$
- Update: $W^{t+1} = W^t + \tilde{U}^t$ where

$$\tilde{U}^t = \begin{bmatrix} 0 & \ldots & 0 \\ & \vdots & \\ 0 & \ldots & 0 \\ \ldots & \frac{\mathbf{1}[y_t = \tilde{y}_t]}{P(\tilde{y}_t)} \mathbf{x}_t & \ldots \\ 0 & \ldots & 0 \\ & \vdots & \\ 0 & \ldots & 0 \\ \ldots & -\mathbf{x}_t & \ldots \\ 0 & \ldots & 0 \\ & \vdots & \\ 0 & \ldots & 0 \end{bmatrix}$$

Row $\tilde{y}_t$

Row $\hat{y}_t$

The Banditron (Kakade, S, Tewari 08)

Theorem

- *Banditron's sample complexity is order of* $\frac{k/\mu^2}{\epsilon^2}$
- *Banditron's runtime is* $O(k/\mu^2)$

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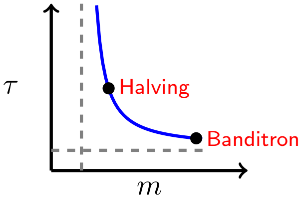
The crux of difference between Halving and Banditron:

- Without having the full information, the version space is non-convex and therefore it is hard to utilize the structure of $\mathcal{H}$
- Because we relied on the Perceptron we did utilize the structure of $\mathcal{H}$ and got an efficient algorithm
- We managed to obtain 'full-information examples' by using exploration
- The price of exploration is a higher regret

Trading samples for runtime

Algorithm	samples	runtime
Halving	$\frac{k^2/\mu^2}{\epsilon}$	$(m+k)^{\tilde{O}(k/\mu^2)}$
Banditron	$\frac{k/\mu^2}{\epsilon^2}$	k/μ^2



Next example: Agnostic PAC learning of fuzzy halfspaces

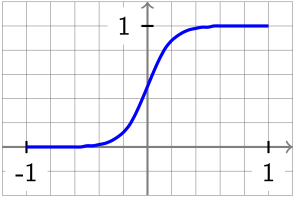
Agnostic PAC:

- $\mathcal{D}$ - arbitrary distribution over $\mathcal{X} \times \mathcal{Y}$
- Training set: $S = (\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m)$
- Goal: use S to find h_S s.t. w.p. $1 - \delta$,

$$\text{err}(h_S) \leq \min_{h \in \mathcal{H}} \text{err}(h) + \epsilon$$

Hypothesis class

$$\mathcal{H} = \{\mathbf{x} \mapsto \phi(\langle \mathbf{w}, \mathbf{x} \rangle) : \|\mathbf{w}\|_2 \leq 1\}, \qquad \phi(z) = \frac{1}{1+\exp(-z/\mu)}$$



- Probabilistic classifier: $\mathbb{P}[h_{\mathbf{w}}(\mathbf{x}) = 1] = \phi(\langle \mathbf{w}, \mathbf{x} \rangle)$
- Loss function: $\mathrm{err}(\mathbf{w}; (\mathbf{x}, y)) = \mathbb{P}[h_{\mathbf{w}}(\mathbf{x}) \neq y] = \left| \phi(\langle \mathbf{w}, \mathbf{x} \rangle) - \frac{y+1}{2} \right|$
- Remark: Dimension can be infinite (kernel methods)

First approach — sub-sample covering

- Claim: exists $1/(\epsilon\mu^2)$ examples from which we can efficiently learn $\mathbf{w}^\star$ up to error of ϵ
- Proof idea:
  - $S' = \{(\mathbf{x}_i, y'_i) : y'_i = y_i$ if $y_i\langle\mathbf{w}^\star, \mathbf{x}_i\rangle < -\mu$ and else $y'_i = -y_i\}$
  - Use surrogate convex loss $\frac{1}{2}\max\{0, 1 - y\langle\mathbf{w}, x\rangle/\gamma\}$
  - Minimizing surrogate loss on $S' \Rightarrow$ minimizing original loss on S
  - Sample complexity w.r.t. surrogate loss is $1/(\epsilon\mu^2)$

Analysis

- Sample complexity: $1/(\epsilon\mu)^2$
- Time complexity: $m^{1/(\epsilon\mu^2)} = \left(\frac{1}{\epsilon\mu}\right)^{1/(\epsilon\mu^2)}$

Second Approach – IDPK (S, Shamir, Sridharan)

Learning fuzzy halfspaces using **I**nfinite-**D**imensional-**P**olynomial-**K**ernel

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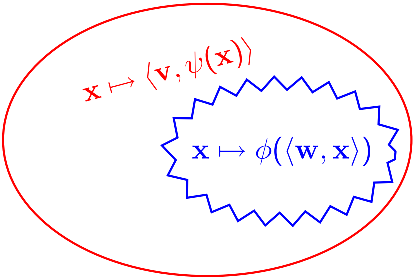
- Main idea: Work with a larger hypothesis class for which the loss becomes convex



Step 2 – Learning fuzzy halfspaces with IDPK

- Original class: $\mathcal{H} = \{\mathbf{x} \mapsto \phi(\langle \mathbf{w}, \mathbf{x} \rangle) : \|\mathbf{w}\| \le 1\}$
- New class: $\mathcal{H}' = \{\mathbf{x} \mapsto \langle \mathbf{v}, \psi(\mathbf{x}) \rangle : \|\mathbf{v}\| \le B\}$ where $\psi : \mathcal{X} \to \mathbb{R}^{\mathbb{N}}$ s.t. $\forall j,\ \forall (i_1, \ldots, i_j),\ \psi(\mathbf{x})_{(i_1,\ldots,i_j)} = 2^{j/2}\, x_{i_1} \cdots x_{i_j}$

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Lemma (S, Shamir, Sridharan 2009)

If $B = \exp(\tilde{O}(1/\mu))$ then for all $h \in \mathcal{H}$ exists $h' \in \mathcal{H}'$ s.t. for all $\mathbf{x}$, $h(\mathbf{x}) \approx h'(\mathbf{x})$.

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Remark: The above is a pessimistic choice of B. In practice, smaller B suffices. Is it tight? Even if it is, are there natural assumptions under which a better bound holds ?
(e.g. Kalai, Klivans, Mansour, Servedio 2005)

Proof idea

- Polynomial approximation: $\phi(z) \approx \sum_{j=0}^{\infty} \beta_j z^j$

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- Therefore:

$$
\begin{aligned}
\phi(\langle \mathbf{w}, \mathbf{x} \rangle) &\approx \sum_{j=0}^{\infty} \beta_j (\langle \mathbf{w}, \mathbf{x} \rangle)^j \\
&= \sum_{j=0}^{\infty} \sum_{k_1,\ldots,k_j} 2^{-j/2} \beta_j 2^{j/2} w_{k_1} \cdots w_{k_j} x_{k_1} \cdot \cdots \cdot x_{k_j} \\
&= \langle \mathbf{v}_{\mathbf{w}}, \psi(\mathbf{x}) \rangle
\end{aligned}
$$

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$$\begin{aligned}\phi(\langle \mathbf{w}, \mathbf{x} \rangle) &\approx \sum_{j=0}^{\infty} \beta_j (\langle \mathbf{w}, \mathbf{x} \rangle)^j \\ &= \sum_{j=0}^{\infty} \sum_{k_1,\ldots,k_j} 2^{-j/2} \beta_j 2^{j/2} w_{k_1} \cdots w_{k_j} x_{k_1} \cdots \cdot x_{k_j} \\ &= \langle \mathbf{v}_{\mathbf{w}}, \psi(\mathbf{x}) \rangle\end{aligned}$$

- To obtain a concrete bound we use Chebyshev approximation technique: Family of orthogonal polynomials w.r.t. inner product:

$$\langle f, g \rangle = \int_{x=-1}^{1} \frac{f(x) g(x)}{\sqrt{1 - x^2}} dx$$

Infinite-Dimensional-Polynomial-Kernel

- Although the dimension is infinite, can be solved using the kernel trick
- The corresponding kernel (a.k.a. Vovk's infinite polynomial):

$$\langle \psi(\mathbf{x}), \psi(\mathbf{x}') \rangle = K(\mathbf{x}, \mathbf{x}') = \frac{1}{1 - \frac{1}{2}\langle \mathbf{x}, \mathbf{x}' \rangle}$$

- Algorithm boils down to linear regression with the above kernel
- Convex! Can be solved efficiently
- Sample complexity: $(B/\epsilon)^2 = 2^{\tilde{O}(1/\mu)}/\epsilon^2$
- Time complexity: m^2

Trading samples for time

Algorithm	sample	time
Covering	$\frac{1}{\epsilon^2\mu^2}$	$\left(\frac{1}{\epsilon\mu}\right)^{1/(\epsilon\mu^2)}$
	$\Uparrow$	$\Downarrow$
IDPK	$\left(\frac{1}{\epsilon\mu}\right)^{1/\mu} \frac{1}{\epsilon^2}$	$\left(\frac{1}{\epsilon\mu}\right)^{2/\mu} \frac{1}{\epsilon^4}$

Summary

- Trading data for runtime (?)
- There are more examples of the phenomenon ....

Open questions:

- More points on the curve (new algorithms)
- Lower bounds ??? Can you help ?