Substitutes, complements and equilibrium in two-sided market models

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Abstract

There are two sets of agents: buyers $\mathcal{B}$ and sellers $\mathcal{S}$. Each type of agent is allowed to trade with as many agents on the opposite side it wishes. Agents' decision process is determined by a market price system p, where $p = (p(s,b),\ (s,b) \in \mathcal{S} \times \mathcal{B})$. Namely, a seller s solves the task $\max_{B \subset \mathcal{B}} [p(s,B) - c(s,B)]$, where $c(s,B)$ is the cost incurred by seller s when he contracts with a set B of buyers. A buyer b, similarly, will solve for $\max_{S \subset \mathcal{S}} [u(S,b) - p(S,b)]$, where $u(S,b)$ is the utility of buyer b after contracting with S sellers.

We examine the existence of competitive equilibrium in this market. We show that equilibria exist in those markets in which all the goods on sale are pure substitutes, or in which all goods are pure complements or finally in which there are appropriate combinations of substitutes and complements. We also establish results about the structure of the sets of equilibrium prices and allocations. We show that the substitution and complementarity requirements are intimately related with the discrete convexity (or concavity) requirements imposed on the corresponding cost and utility functions of the market agents.

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1. Introduction

We consider here a two-sided market, in which agents are divided in two complementary groups, say, sellers $\mathcal{S}$ and buyers $\mathcal{B}$. We allow here both that a seller trade with a few buyers and that a buyer trade with a few sellers. Trades between agents belonging to a same group are forbidden. A trade between a seller s and a buyer b consists in the transfer (from s to b) of some given item (s,b) (and, possibly, of a transfer of money from b to s). This market is a market with "indivisible" goods since a trade between a pair of agents involves at most a single item[1] (either agents conclude a deal or they do not). We shall assume here that utility is transferable, and that money transfers between agents are allowed.

In this paper, we study the issue of existence of competitive equilibrium and investigate the structures of the sets of equilibrium prices and allocations. To talk about competitive outcomes in this market, we shall have to assume that for every item potentially transferred from s to b a price $p(s,b)$ emerged. This price system $p=(p(s,b))$ determines buyers' demand schedules, on the one hand, and sellers' supply schedules, on the other. We define a competitive equilibrium in the standard way.

In markets with indivisible goods, equilibria need not to obtain in the many-to-many case in contrast to the one-to-one setup. We give conditions to warrant existence of competitive equilibrium. Roughly speaking, we require some kind of "discrete convexity" of utilities. Namely, we show that there exist competitive equilibria in markets in which all goods are pure substitutes, or in which all goods are pure complements, or finally in which there are specific combinations of substitutes and complements.

Let us now briefly recall some salient aspects of two-sided market models and discuss within this setup the substitutes and complements cases.

The early studies of two-sided markets models focused on one-to-one setups: - the assignment problem (Koopmans and Beckmann (1957), Shapley and Shubik (1972)) and - the marriage matching problem (Gale and Shapley (1962)). In both these setups, no requirements were needed for the existence of solutions, moreover sets of solutions exhibited lattice structures. As we shall point out later on, the gross substitution property was automatically satisfied in this setup.

In a one-to-many (even a many-to-many) framework, Crawford and Knoer (1981) proved existence of equilibrium imposing the separability of utility or production functions under capacity constraints. Kelso and Crawford (1982) introduced the gross substitution property to establish existence in the one-to-many subcase. Gul and Stacchetti (1999) proposed two conditions equivalent to the gross substitution property. They claim that gross substitution is necessary and sufficient to ensure existence of Walrasian equilibria. This is a slight overstatement. As it turns out, Danilov, Koshevoy, and Murota (1998) show that discrete concavity is the appropriate condition. Moreover, gross substitution is one particular instance of discrete concavity. We clarify this relationship in the present work. We show first that GS-functions (in a boolean context[2]) are nothing else than polymatroidal (PM) concave functions. If we

[1] A more general setup might be one in which trades between opposed agents would involve a few commodities, possibly divisible.

[2] We call boolean the context in which agents are allowed to consume or produce no more than one item of

allow agents to consume or produce eventually more than one item of a given type, then the GS condition turns out to be weaker than the PM condition and yet too weak for existence.

The case of pure complements also has a respectable tradition in the economic literature and it turns out it is also an instance of discrete concavity. We shall not dwell too deeply upon this issue now. But we can recall, for instance, that Samuelson (1947) associated the idea of complementarity with that of supermodularity (see Topkis (1978)). We follow this practice. Some time ago Danilov, Koshevoy, and Sotskov (1994, 1997) relied upon sub/supermodularity conditions to prove the existence of equilibria in an economy with intellectual (or informational) goods.

In the present work, we shall also establish existence in a mixed case in which the market goods are partitioned in two groups. The goods of the first group are mutual substitutes, and the goods of the second are mutual complements. It will be important here that buyers (consumers) and sellers (producers) have consensual views about this partition in the sense that they all agree about which goods are substitutes and which are complements. This requirement will be called the Compatibility Principle. In the job market model of Kelso and Crawford (1982), the Compatibility Principle will spell as follows. The workers perceive the firms as substitutes, thus by compatibility, firms should perceive the workers as substitutes as well.

2. A two-sided market model

Let there be two (finite) sets of agents $\mathcal{S}$ and $\mathcal{B}$. We shall call them *sellers* and *buyers*[3]. Each agent is allowed to form a partnership (or a deal) with agents from the opposite side. Moreover, an agent can have as many partners as he wants. In other words, we are in a many-to-many setup. We introduce the following definition which incidentally accounts for this multiplicity of possible deals.

Definition A *matching* is an arbitrary subset $\mu \subset \mathcal{S} \times \mathcal{B}$. Denote by $\mu(b) = \{s,\ (s,b) \in \mu\}$ and $\mu(s) = \{b,\ (s,b) \in \mu\}$.

An elementary deal consists in a transfer of *at most* one item (s,b) from the seller s to the buyer b[4]. When buyer b gets hold of a set $S \subset \mathcal{S}$ of items, his utility increases by the amount $u(S,b)$. Similarly, when seller s assigns his items to a set $B \subset \mathcal{B}$ of buyers, he experiences a loss of utility. In our opinion, it is convenient to view sellers as producers. The production cost of a set B of items for s is denoted by $c(s,B)$.

a given type.

[3]Of course, we can think of them as "firms and workers" (like in Kelso and Crawford (1982) or Roth (1984)), or as "colleges and students" (like in Gale and Shapley (1962)), or, if we allow ourselves to pursue the allegory of marriage, as "men and women" (though, in promiscuous marriages), finally, as "service-providing facilities and customers".

[4]One can assume that the goods offered by a given seller are somewhat similar: one seller supplies houses, another - cars, the third planes, and so on.... Furthermore, we assume that each buyer needs no more than one item of any given type.

It seems rather natural to pose both that $c(s,\emptyset)=0$ and $u(\emptyset,b)=0$. We define now the total gain of matching μ to be equal to the sum of buyers' gains minus the sum of sellers' production costs,

$$v(\mu)=\sum_b u(\mu(b),b)-\sum_s c(s,\mu(s)).$$

Definition A *core outcome* (or *stable outcome*) is a pair (μ,x), where μ is some matching and $x: N=\mathcal{S}\cup\mathcal{B}\to\mathbb{R}$ is a vector of agents utilities, which satisfy the following conditions:

1. $v(\mu)=x(N)$,
2. For any coalition $K=(S\cup B)\subset N$ and for any matching $\mu'\subset S\times B$, we have $x(K)\geq v(\mu')$.

We define $x(K)=\sum_{k\in K}x(k)$. In particular, $x(i)\geq 0$ for each agent $i\in N$.

Example 1 *Empty core with heterogeneous buyers.* We consider a market with two sellers s,s' and two buyers b,b'. Seller s has two bottles of gin g,g' on sale. Seller s' has two bottles of tonic water t,t' on sale. The cost functions of s (resp. s') are:

$$\begin{aligned} c(s,g)=c(s,g')=0, &\qquad c(s,\{g,g'\})=+\infty \quad (\text{or } 100),\\ c(s',t)=c(s',t')=0, &\qquad c(s',\{t,t'\})=+\infty \quad (\text{or } 100). \end{aligned}$$

In this example, the items produced by anyone of the sellers are perfect substitutes with respect to production. Moreover each seller has a capacity constraint: he can sell (and produce) at most one bottle.

The utility functions of the buyers are:

$$\begin{aligned} u(b,g)=u(b,t)=0, &\qquad u(s,\{g,t\})=1,\\ u(b',g')=u(b',t')=1, &\qquad u(b',\{g',t'\})=1. \end{aligned}$$

We notice that b likes to consume gin and tonic together and thus for him these goods are complements. The second buyer b' will be happy to drink just anything. He is indifferent with respect to gin, tonic, and *gin & tonic.*

Easy computations show that: $v(\{s,s',b\})=1$ [a], $v(\{s,b'\})=1$ [b], $v(\{s',b'\})=1$ [c] and $v(\{s,s',b,b'\})=1$ [d].

Suppose that x is a payoff vector in the core. Then from [a] and [d], we have $x(b')=0$. From [b] and [d], we have $x(s')=x(b)=0$. From [c] and [d], we have $x(s)=x(b)=0$. Thus $x\equiv 0$. This contradicts the Pareto optimality of x.

Example 2 *The separable case.* In the separable (or additive) case, seller s's cost function $c(s,B)$ is given as the sum $c(s,b)$ over $b\in B$ (we define the utility of a buyer b in a similar fashion). Any trade is then decomposed in a series of separate and elementary trades (s, b). Now, clearly, a trade (s, b) obtains as soon as $u(s, b) \geq c(s, b)$; the buyer b then transfers the amount of money $p(s, b)$ to the seller s, and $u(s, b) \geq p(s, b) \geq c(s, b)$.

Each deal (s, b) is concluded at a price $p(s, b)$. We can always assume that the price $p(s, b)$ of an item (s, b) would lie in between $u(s, b)$ and $c(s, b)$ even if the deal (s, b) does not finally materialize. Thus $p = (p(s, b))$ is the market price system at equilibrium. The stable outcomes coincide with the competitive allocations at equilibrium in this separable case.

It is now time to define what is a competitive equilibrium in this setup. Let there be a price system $(p(s, b),\ s \in \mathcal{S},\ b \in \mathcal{B})$, in our market. The seller s solves the following problem

$$\max\,[p(s, B) - c(s, B)], \quad \text{for} \quad B \subset \mathcal{B}, \tag{1}$$

while the buyer b solves for

$$\max\,[u(S, b) - p(S, b)], \quad \text{for} \quad S \subset \mathcal{S}. \tag{2}$$

We have an equilibrium when the solutions to (1) and (2) are consistent. More precisely,

Definition An *equilibrium* is a pair (μ, p), where μ is a matching and $p = (p(s, b))$ is a price system such that $\mu(s)$ solves (1), for each $s \in \mathcal{S}$, and $\mu(b)$ solves (2), for each $b \in \mathcal{B}$.

The competitive allocations belong to the core. Precisely, given a pair (μ, p), we define the utility $x(s)$ of seller s to be $x(s) = p(s, \mu(s)) - c(s, \mu(s))$ (and similarly for buyer b).

Proposition 1. *Given an equilibrium (μ, p), the pair (μ, x) is a stable outcome.*

The proof is standard, thus omitted.

The converse of this Proposition is not true as shown in the following example.

Example 3 *Core without equilibria.* Again there are two sellers and two buyers. The buyers are as in Example 1: the first buyer desires gin and tonic together, the second will be happy with either. However the cost functions are different. Now the production cost of the first bottle is equal to 1, while the second bottle is produced at no cost. This context obtains in the case of informational goods in which typically the cost of producing the first specimen is significantly larger than that of duplicates (Danilov, Koshevoy, and Sotskov (1994)).

It is easy to see that $v(K) = 0$ for any coalition K. Thus the core consists of the unique point $(0, 0, 0, 0)$.

Let us show that there is no equilibrium. Suppose that the equilibrium prices of seller s's items are equal to p, p' and that those of seller s' are equal to q, q'. Since an equilibrium allocation is stable, it yields a net surplus of 0 to every agent on the market. We have therefore the following system of inequalities:

$$\begin{array}{llll}
1. & p \leq 1, & p' \leq 1, & p + p \leq 1 \\
2. & q \leq 1, & q' \leq 1, & q + q \leq 1 \\
3. & 0 \leq p, & 0 \leq q, & 1 \leq p + q \\
4. & 1 \leq p', & 1 \leq q', & 1 \leq p' + q'.
\end{array}$$

The inequalities $0 \leq p$, $1 \leq p'$ and $p + p' \leq 1$ give $p = 0$ and $p' = 1$. Similarly, $0 \leq q$, $1 \leq q'$ and $q + q' \leq 1$ give $q = 0$ and $q' = 1$. However this is never compatible with $1 \leq p + q$.

Examples 1 and 3 show that, in contrast to the one-to-one setup, equilibria need not obtain in the many-to-many setup. In the sequel, we provide the conditions which warrant existence of equilibrium (and, consequently, of stable outcomes[5]). We consider two main conditions. The first is the gross substitution condition as introduced by Kelso and Crawford (1982)[6]. The second is the polar (in some sense) condition of complementarity. We shall consider the case of substitutes and that of complements separately and then briefly discuss a mixed case. We start with a few generalities about the existence and the structure of the set of equilibria in the transferable setup.

3. Generalities about equilibria

In the transferable case, the existence issue is related to an optimization problem. To this end, we aggregate all buyers into a single consumer and all sellers into a single producer.

Namely, let $\Omega = \mathcal{S} \times \mathcal{B}$. Consider the following two function U and C on the set 2^{Ω}:

$$U(\mu) = \sum_b u(\mu(b), b), \qquad C(\mu) = \sum_s c(s, \mu(s)),$$

where $\mu \subset \Omega$ is a matching. $U(\mu)$ is the aggregate utility derived from μ, and $C(\mu)$ is the aggregate production cost of μ. We are interested in the aggregate surplus $U - C$. The maximum of this function is equal to $v(N)$. The set of optimal matchings is denoted by $\mathbf{M}$. Proposition 1 states that any equilibrium matching belongs to $\mathbf{M}$. An optimal matching will be a competitive allocation, whenever it can be supported by some price system p.

A price p *supports* a matching μ^* with respect to consumption if $U(\mu) - U(\mu^*) \leq p(\mu) - p(\mu^*)$. That is: p is a supergradient to U at the point μ^*. Similarly, a price p

[5]The conditions under which stable outcomes can be decentralized remain yet to be found. Some results were obtained by Kaneko (1982), Quinzii (1984) and Kelso and Crawford (1982).

[6]Of course, the notion of gross substitutability was formulated and investigated long ago. (One can trace it back to Hicks. Morishima, Negishi, Arrow and Hahn, and many others made important contributions.) The classical definition has been formulated in the case of single-valued demands. In economies with indivisible commodities, demands at certain prices are unavoidably multi-valued. Thus one has to provide for an appropriate formulation of this condition. Note that Polterovich and Spivak (1982) proposed an alternative formulation to that of Kelso and Crawford (1982).

supports a matching μ^* with respect to production if $C(\mu) - C(\mu^*) \geq p(\mu) - p(\mu^*)$. That is: p is a subgradient to U at the point μ^*. Of course, p supports μ^* with respect to consumption and production if and only if p separates the functions $U - U(\mu^*)$ and $C - C(\mu^*)$, that is,

$$U - U(\mu^*) \leq p - p(\mu^*) \leq C - C(\mu^*).$$

Proposition 2 *A pair (μ^*, p) is an equilibrium if and only if p separates the functions $U - U(\mu^*)$ and $C - C(\mu^*)$.*

Proof Suppose that (μ^*, p) is an equilibrium. Then, for every buyer b, the following inequality holds,

$$u(\cdot, b) - p(\cdot, b) \leq u(\mu^*(b), b) - p(\mu^*(b), b).$$

Adding these inequalities, we obtain $U(\cdot) - p(\cdot) \leq U(\mu^*) - p(\mu^*)$. Similarly for C.

Conversely, suppose that p supports μ^* with respect to consumption, thus $U(\cdot) - U(\mu^*) \leq p(\cdot) - p(\mu^*)$. This inequality is equivalent to the following series of inequalities

$$u(\cdot, b) - u(\mu^*(b), b) \leq p(\cdot, b) - p(\mu^*(b), b), \qquad b \in \mathcal{B}.$$

Each inequality implies that $\mu^*(b)$ is optimal for buyer b at the price $p(\cdot, b)$. Similarly for the sellers. ■

Corollary *The set of equilibria has the form $\mathbf{M} \times \mathbf{P}$, where $\mathbf{P}$ is the set of prices separating the functions U and $C + v(N)$.*

The set $\mathbf{P}$ can be empty, as in Example 1 and 3. In order to exhibit a linear functional p separating the "osculating" functions U and $C + \max(U - C)$ (that is in order to exhibit a "sandwich") some kind of "concavity" requirement is needed for the functions U and $-C$. Danilov and Koshevoy (2000) (and, see also Danilov, Koshevoy, and Murota (1998)) develop this point at length. We give here some flavor of their work, restraining ourselves to functions given on the boolean lattice 2^Ω.

First, let us identify the set 2^Ω with the set $\{0,1\}^\Omega$ in the vector space $\mathbb{R}^\Omega$. To this end, we associate the characteristic vector $\mathbf{1}_A : \Omega \to \mathbb{R}$,

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A. \end{cases}$$

to every subset $A \subset \Omega$.

The functions U and C are now defined on the set of vertices of the unit cube $Q := [0,1]^\Omega = \mathrm{co}(\{0,1\}^\Omega$. Denote by $\mathrm{co}(U)$ the concavification of U, i.e., the minimal concave function on the cube Q which is superior or equal to U. Obviously, $\mathrm{co}(U)(X) = U(X)$ at every vertex X of Q. A price system p is viewed, thus, as the linear functional on the space $\mathbb{R}^{\Omega}$, which is equal to $p(\omega)$ at the point $\mathbf{1}_{\omega}$. Now to any linear functional $p \in (\mathbb{R}^{\Omega})^{*}$, we associate the convex set,

$$D(U,p) := \mathrm{Argmax}\{\mathrm{co}(U) - p\}$$

in the cube Q. It is an integer polytope. (A polytope is integer if its vertices are integer.) This polytope will be called a *cell* (or an *affinity area*) of the function U [7]. The cell $D(U,p)$ is the set of points of Q, for which the function $\mathrm{co}(U)$ coincides with the affine function $p + \max(U - p)$.

Lemma *If $D(U,p)$ is a cell of U, and D' is a face of $D(U,p)$, then D' is also a cell of U.*

Proof The face D' is the set of points in $D(U,p)$, where a certain linear functional q on $\mathbb{R}^{\Omega}$ attains its maximum. Then $D' = D(U, p - \varepsilon q)$, where ε is a small and positive number. ■

The following proposition shows that the crucial reason for the existence of a linear functional separating U and C ("a sandwich") is that the intersections of cells of the functions U and $-C$ are integer polytopes.

Proposition 3 (Sandwich) *Let U and C be functions on 2^{Ω}, and let $C \geq U$. Suppose that, for every p, the intersection of the polytopes $D(U,p)$ and $D(-C,p)$ is an integer polytope. Then U and C are separated by some linear functional.*

Proof Since $C \geq U$, we have $U - C \leq 0$. We prove the following stronger claim.

Claim $\mathrm{co}U + \mathrm{co}(-C) \leq 0$.

Let x be an arbitrary point in the cube Q. Let D be a cell of U containing x, and D' be a cell of $-C$ containing x as well. The intersection $D \cap D'$ is an integer polytope by assumption. Let $X_1, \cdots, X_n$ be the vertices of this polytope. Then x is obtained as a convex combination of these vertices, $x = \sum_i \alpha_i X_i$, where $\alpha_i \geq 0$, and $\sum_i \alpha_i = 1$. We assert that $\mathrm{co}(U)(x) = \sum_i \alpha_i U(X_i)$ and $\mathrm{co}(-C)(x) = -\sum_i \alpha_i C(X_i)$. We prove the first equality (the second is obtained similarly).

Since $\mathrm{co}(U)$ is affine on D and $X_1, \ldots, X_n \in D$, then $\mathrm{co}(U)(x) = \sum_i \alpha_i \mathrm{co}(U)(X_i)$. However $\mathrm{co}(U)$ coincides with U at the vertices of the cube Q, therefore $\mathrm{co}(U)(X_i) = U(X_i)$ and $\sum_i \alpha_i U(X_i) = \mathrm{co}(U)(x)$.

Now,

$$\mathrm{co}(U)(x) + \mathrm{co}(-C)(x) = \sum_i \alpha_i U(X_i) - \sum_i \alpha_i C(X_i) = \sum_i \alpha_i [U(X_i) - C(X_i)] \leq 0.$$

[7] Actually it is a cell of $\mathrm{co}(U)$. For brevity, we write "cell of U".

This terminates the proof of our claim.

Let us return to the proof of Proposition 3. We have the following inequality $\mathrm{co}(U) + \mathrm{co}(-C) \leq 0$, that is $-\mathrm{co}(-C) \geq \mathrm{co}(U)$.

The function on the left side of this inequality is convex, while that on the right side is concave. By a classical separation of convex sets argument, there exists both a linear function q and a real α such that

$$-\mathrm{co}(-C) \geq q + \alpha \geq \mathrm{co}U.$$

Since $\mathrm{co}(U) \geq U$, $q(X) + \alpha \geq U(X)$ for any integer point X of Q. Similarly, $C(X) \geq q(X) + \alpha$, for any integer point X in Q. ■

Remark 1. If, in Proposition 3, the function U is monotone, then there exists a monotone separating functional p. To start with, note that $\mathrm{co}(U)$ is a monotone function on the cube Q. Now instead of considering $\mathrm{co}(U)$, we consider its monotone extension F on the whole positive orthant $\mathbb{R}_+^\Omega$. It is defined as follows: $F(x) = \mathrm{co}(U)(\min(x, \mathbf{1}_\Omega))$. F is concave and is everywhere (in the cube Q) inferior or equal to $-\mathrm{co}(-C)$. In this case, any separating functional p will be non-negative.

Remark 2. The theory of Discrete Concavity (Danilov and Koshevoy (2000)) characterizes the classes of discrete functions for which Proposition 3 holds true. In the next section, we show that *GS*-functions form a class of discrete concavity on the boolean cube.

4. The Gross Substitution Property

Let Ω be an arbitrary finite set of items. We identify here a *bundle* to a subset $A \subset \Omega$. A utility function is a mapping $u : 2^\Omega \to \mathbb{R}$ for which $u(\emptyset) = 0$. A price functional is the simplest example of such a map. Let $p : \Omega \to \mathbb{R}$ represent a price schedule. Then the value of a bundle A for the price schedule p is given by $p(A) = \sum_{a \in A} p(a)$.

Given a utility function u and a price p, the *net utility* derived from A is defined by $u(A) - p(A)$. A consumer with utility u selects the bundles yielding the highest possible net utility at prices p. Denote by $D(u,p)$ the set of optimal bundles, that is:

$$D(u,p) = \{A \subset \Omega, \quad u(A) - p(A) \geq u(A') - p(A') \qquad \text{for any } A' \subset \Omega\}.$$

$D(u,p)$ is the demand set at the price p. (Henceforth, we shall drop the parameter u when no confusion is possible.)

Definition (Kelso and Crawford) The utility function u is said to *generate gross substitutable demands* or (in short) is a *GS-function* if, for any pair of prices p, q, such that $q \geq p$, and for any $A \in D(p)$, there exists $B \in D(q)$ such that $\{\omega \in A, p(\omega) = q(\omega)\} \subset B$.

In other words: suppose that in the process of going from the price system p to the price system q, some prices increase, while others remain unchanged. If some item ω was demanded at prices p, and if ω's price stayed put in q, then it might be demanded at prices q.

One can expect that this property hold true when the items in Ω are substitutable. Intuitively, this excludes the complementarity-type relationship between goods, that is evidenced in such commodity bundles as the bundle "key and key-lock" or the bundle "left and right glove". Incidentally Gul and Stacchetti (1999) formalize the absence of complementarity between goods in their Single Improvement property.

Definition (Gul and Stacchetti) Suppose $A \notin D(p)$, hence A does not maximize $u-p$. Then there exists a better bundle B (with $u(B)-p(B) > u(A)-p(A)$), such that $|A-B| \leq 1$ and $|B-A| \leq 1$. A utility function u which satisfies this condition for all p is said to have the *SI-property*.

In other words, if the bundle A is not optimal at prices p, then a better bundle B can be found, which is derived from A by performing any of the following three operations: removing an item from A, or adding an item to A, or finally doing both. Thus we may improve upon a bundle by performing these elementary operations.

Gul and Stacchetti (1999) in their Theorem 1 prove that the *GS* and *SI* properties are equivalent. We, in turn, propose an alternative characterization of the *GS*-property. To this end, we introduce the following notions.

Definition (i) A *root* in the space $\mathbb{R}^\Omega$ is any of the vectors $\mathbf{1}_a, \mathbf{1}_a - \mathbf{1}_b$, where $a, b \in \Omega$.

(ii) A convex polytope in the space $\mathbb{R}^\Omega$ is a *g-polymatroid* if any of its edges is parallel to some root.

(iii) A function u, defined on the set 2^Ω, is called *PM-concave* if its cells are g-polymatroids.

Theorem 1 *A function on the set* 2^Ω *is a* GS*-function if and only if it is* PM*-concave.*

Proof Suppose that u is a *GS*-function. Let us check that each edge of a cell of u is parallel to some root, and this for each cell. Consider a given edge. According to Lemma 1, we can assume that this edge is some cell $D(\mathrm{co}(u), p)$ of the function $\mathrm{co}(u)$. Let this edge connect the two vertices $\mathbf{1}_A$ and $\mathbf{1}_B$ of the cube Q. Then the bundles A and B are the only optimal bundles at the price p. As $B \neq A$, we can assume that there exists $b' \in B \setminus A$. Let us slightly increase the price of item b'. Then at this resulting price system p', bundle A remains the sole optimal bundle. B which isn't optimal any more, is nevertheless preferred to any other non-optimal bundle. By the *SI*-property, bundle A can be obtained from bundle B provided one performs one of the following elementary operations: removing some item b, or removing some b and adding some a. This means that the vectors $\mathbf{1}_A$ and $\mathbf{1}_B$ differ by a root.

Conversely, suppose u is a *PM*-function. Let us check that the *SI* property holds. Let A be a suboptimal bundle at prices p. Given a root r, denote by $d(r)$ the derivative of function $\mathrm{co}(u) - p$ at the point A in the direction r. By *PM*-concavity of $\mathrm{co}(u)$, $d(r) > 0$, for some root r. This is because the edges of the cells of $\mathrm{co}(u)$ are parallel to roots.

Inequality $d(r) > 0$ implies that the (open) segment $(A, A+r)$ intersects some cell D of u. Since, both the (closed) segment $[A, A+r]$ and D are integral g-polymatroids, by the Edmonds-Frank theorem (below), their intersection is an integral polytope namely $[A, A+r]$. Now this implies that the function $\mathrm{co}(u) - p$ is affine on this segment. In particular, the value of this function at the point $B := A + r$ is larger than its value at the point A. ■

Remark 1. The functions considered above, were given on the vertices of the unit cube. But it would be more appropriate to develop the theory of GS and PM functions for functions defined on the whole of the lattice $\mathbb{Z}^\Omega$ of integer points of $\mathbb{R}^\Omega$. We expose this theory in Danilov, Koshevoy, and Lang (2001).

Remark 2. Up to now, following in this Gul and Stacchetti (1999), we considered only those functions on 2^Ω, whose values were finite. But there are good reasons to consider functions which may take infinite values. In particular, when dealing with "utility" functions, it is convenient to consider the value $-\infty$. Similarly when we are dealing with "cost functions", we'd like to consider the value $+\infty$. Of course, the cells of u, in this case, will cover the convex hull of its effective domain $\mathrm{co}(\mathrm{dom}(u))$, and not the whole cube Q. Theorem 1 still holds in this case.

Theorem 2 (Sandwich theorem) *Let U and $-V$ be PM-concave functions on $\{0,1\}^\Omega$. Suppose that $V \geq U$. Then there exists a linear functional p and a real number α such that $V \geq p + \alpha \geq U$.*

This result (proven first by Murota (1996)) follows from both Proposition 3 and the polymatroid intersection theorem given below (see, for instance, Frank and Tardös (1988)).

Theorem (Edmonds-Frank) *The intersection of two integer g-polymatroids* [8] *is an integer polytope.*

5. Markets with pure substitutes

Let us return to our two-sided market whose sellers $s \in \mathcal{S}$ have cost functions $c(s, \cdot)$ on $2^{\mathcal{B}}$, and whose buyers $b \in \mathcal{B}$ have utility functions $u(\cdot, b)$ on $2^{\mathcal{S}}$. We impose the following assumption.

Assumption (Gross Substitution) *$u(\cdot, b)$ is a* GS*-function for every buyer $b \in \mathcal{B}$ and $-c(s, \cdot)$ is a* GS*-function for every seller $s \in \mathcal{S}$.*

Theorem 3 *A two-sided market in which the Gross Substitution Assumption is satisfied has equilibria.*

Proof We prove existence by an aggregation argument. Let U and C be the aggregate utility and cost functions on 2^Ω, where $\Omega = \mathcal{S} \times \mathcal{B}$. We assert that U and $-C$ are both GS-functions. Indeed,

[8] Note that Edmonds and Frank define a g-polymatroid as a polytope given by specific linear inequalities (see Frank and Tardös (1988)). Their definition and the one given above are equivalent (see Danilov and Koshevoy (2000)).

$$D(U,p) = \prod_b D(u(\cdot,b), p(\cdot,b)),$$

and similarly for $-C$. It is straightforward to check that $D(U,p)$ satisfies the gross substitution property. It is done either by referring to the definition or by checking that the edges of the associated polytope $\mathrm{co}(D(U,p)) = \prod_b \mathrm{co}(D(u(\cdot,b),(p(b)))$ have the required form.

Now all follows from Proposition 2, Theorem 1 and Theorem 2. ∎

We are now ready to state two assertions about the structures of the set of equilibrium prices and of the set of equilibrium allocations. From Corollary 1, we know that the set of equilibria has the form $\mathbf{M} \times \mathbf{P}$. If $\mu^* \in \mathbf{M}$ and $p^* \in \mathbf{P}$ then

$$\begin{array}{rcl} \mathbf{M} & = & \mathrm{Argmax}(U - p^*) \cap \mathrm{Argmax}(p^* - C), \\ \mathbf{P} & = & \partial \mathrm{co} U(\mu^*) \cap -\partial \mathrm{co}(-C)(\mu^*). \end{array}$$

Here ∂ denotes the superdifferential of a concave function. Recall that the superdifferential of a function at a point is the set of all supergradients to this function at this point.

Moreover the sets $\mathbf{M}$ and $\mathbf{P}$ are endowed with specific structures.

Definition We say that a subset $M \subset 2^\Omega$ is *in-between-convex* set if, given any two elements μ^* and μ^{**} such that $\mu^* \subset \mu^{**}$, it contains all intermediate μ as well (that is it contains all μ such that $\mu^* \subset \mu \subset \mu^{**}$)[9].

Theorem 4 *In a two-sided market satisfying the Gross Substitution Assumption,*

a) $\mathbf{M}$ *is an in-between-convex subset of* 2^Ω *and*

b) $\mathbf{P}$ *is a sublattice of* $\mathbb{R}^\Omega$.

Proof a) Since the intersection of in-between-convex subsets is in-between-convex, it suffices to prove that $\mathrm{Argmax}(U - p)$ is in-between-convex. The function $U - p$ is a GS-function, thus is submodular (Gul and Stacchetti (1999), Lemma 6). Then to get assertion a) it suffices to prove that the set of maxima of a submodular function f is in-between-convex.

Indeed, let μ^* and μ^{**} be two maxima of f and let $\mu^* \subset \mu \subset \mu^{**}$. Denote by $\mu' = (\mu^{**} - \mu) \cup \mu^*$. Clearly, $\mu \cap \mu' = \mu^*$ and $\mu \cup \mu' = \mu^{**}$. Then, on the one hand,

$$f(\mu) \le f(\mu^*) = f(\mu^{**}) \quad \text{and} \quad f(\mu') \le f(\mu^*) = f(\mu^{**}).$$

On the other hand, by submodularity,

$$f(\mu) + f(\mu') \ge f(\mu \cap \mu') + f(\mu \cup \mu') = f(\mu^*) + f(\mu^{**}) \ge f(\mu) + f(\mu').$$

[9] Analogously, this notion can be formulated for any ordered set.

The inequalities hold with equality. In particular, μ also belongs to $\mathrm{Argmax}(f)$.

$b)$ The intersection of sublattices is a sublattice, thus we need only prove that $\partial f(x)$ is a sublattice for every PM-concave function f. Without loss of generality, we may assume that $x = 0$ and that the function f is homogeneous (of degree one). One-dimensional cells of this function, i.e. rays, are generated by roots. Hence the superdifferential is given by a system of linear inequalities of the form $c(a) \leq p(a) \leq c'(a)$ or of the form $c(a, b) \leq p(a) - p(b) \leq c'(a, b)$, where $c(a), c'(a), \ldots$ are constants. Every inequality of this kind defines a sublattice, as well as the whole system. ∎

To conclude this section, we discuss a few examples of GS-functions (or alternatively of PM-concave functions).

Example 4 Consider a seller which can not produce more than one item, i.e., his cost $c(B)$ equals $+\infty$ as soon as B contains more than one element. Then $-c$ is PM-concave because its cells are g-polymatroids. Indeed, the cells of c are faces of the unit simplex, whose vertices are constituted by 0 and the basis vectors $\mathbf{1}_b$, for $b \in \mathcal{B}$.

If all sellers are of this type (as are the workers in Kelso and Crawford (1982)) the gross substitution property is fulfilled automatically on the sellers' side. The market economy has competitive equilibria if moreover we require gross substitution on the buyers' side.

Example 5 Consider now a buyer who is not eager to consume more than one item. His utility then takes the form,

$$u(S) = \max_{s \in S} u(s).$$

The function u is PM-concave. This holds because the monotone extention of a PM-concave function is PM-concave.

Therefore the gross substitution property is automatically fulfilled on the buyers' side if all buyers are of this type. If, on top, we require gross substitution on the sellers' side, we have existence. Note that Kaneko (1982) considered the particular subcase in which cost functions were additive.

Observe, finally, that if sellers cannot produce more than one item and buyers have no need for more than one item, then the gross substitution property is fulfilled automatically on both sides of the market and clearly competitive equilibria always obtain in this kind of two-sided market. Actually this is the market investigated by Shapley and Shubik (1972).

Example 6 Any separable (or additive, or linear) function on 2^{Ω} is a GS-function. This explains why the market in Example 2 has competitive equilibria.

We can devise a more interesting instance in which separable functions are used, like in Crawford and Knoer (1981). They consider separable cost functions with capacity constraints. Let l be a linear function and k capacity constraint. A cost function with capacity constraint is a function c which coincides with l as long as $|B| \leq k$, and is equal to $+\infty$ elsewhere.

(Example 4 can be viewed as a special case of capacity constraint, where the capacity is $k = 1$.) Crawford and Knoer (1981) proved existence of equilibria in this context. Of course, their result is clarified when one realizes that such functions are PM-concave. (More generally, any PM-concave function to which we impose a capacity constraint remains a PM-concave function.)

Example 7 Bevia, Quinzii, and Silva (1999) considered another interesting case. They showed that a function of the form $\sum_b u_b(x_b) + \phi(\sum_b x_b)$, where ϕ is a concave function of one variable, is a GS-function. When $\phi(t)$ is equal to 0 for $t \leq k$ and is equal to $-\infty$ for $t > k$, we are back to the discussion in Example 6.

The previous examples we considered, are particular cases of a more general construction (see Danilov, Koshevoy, and Murota (1998) or Danilov and Lang (2000)) which we describe now. Suppose $\mathcal{T}$ is a family of subsets of Ω. This family is called *laminar* if, for every $A,\ B \in \mathcal{T}$, one of the three conditions hold: $A \subset B$, $B \subset A$ or $A \cap B = \emptyset$. Consider now the collection of concave functions of one variable ϕ_A, indexed by $A \in \mathcal{T}$. We construct the new function U on the boolean set 2^Ω defining, for $X \subset \Omega$, that

$$U(X) = \sum_{A \in \mathcal{T}} \phi_A(|X \cap A|).$$

Then U is a PM-concave function.

One can also use the fact that the (infimal) convolution of PM-concave functions is a PM-function to derive new PM-concave functions. Other, in order to check the GS property, one can use the following characterization: u is GS iff its Fenchel conjugate u^* is supermodular (Danilov and Lang (2000)).

6. Markets with pure complements

We consider now the polar case to a market with substitutes, that of a market with pure complementary goods. Recall that if u is a GS-function, then it is submodular and thus has "decreasing marginal utility". Conversely, a supermodular function u has "increasing marginal utility". This means that the difference

$$u(A) - u(A \setminus a)$$

is a monotone function of $A \subset \Omega$, which means that the increment of utility derived from adding an item a to a bundle A is greater the larger the bundle A. This property of the utility function points out to the existence of some complementarity among the goods added-to-the-bundle and those entering-the-bundle. We require submodularity of cost functions to model complementarity in the production processes.

We impose here the following assumption.

Assumption (Complementarity) $u(\cdot, b)$ *is a supermodular function for every buyers* $b \in \mathcal{B}$ *and* $c(s, \cdot)$ *is a submodular function for every seller* $s \in \mathcal{S}$.

Theorem 5 *Consider a two-sided market in which the Complementarity assumption is satisfied. Then*

a) this market has equilibria,

b) the set $\mathbf{M}$ *of equilibrium allocations is a sublattice of* 2^{Ω} *,*

c) the set $\mathbf{P}$ *of equilibrium prices is an in-between-convex subset of* $\mathbb{R}^{\Omega}$.

Proof The proof is very similar to that which was developed in the pure substitutes case. The aggregate utility function U is clearly supermodular, while the aggregate cost function C is submodular. The appropriate version of Proposition 3, which states the existence of a separation between supermodular and submodular functions, used here, is due to Frank (1982) (or else see Danilov, Koshevoy, and Sotskov (1994)). The existence of a separation again rests on the shape of the cells of the relevant functions. We only discuss the matter for the supermodular function U . Its concavification $\mathrm{co}(U)$ is linear over simplexes of the standard triangulation of the unit cube $Q = [0,1]^{\Omega}$. Namely, let π be a weak order on the set Ω . The corresponding simplex $\sigma(\pi)$ consists of all monotone maps $(\Omega, \pi) \to [0,1]$. These simplexes $(\sigma(\pi))$ constitute the standard simplicial decomposition of the cube Q when π runs through the set of all weak orders on Ω . (The same holds true for the function $-C$.) The intersection of any of these simplexes is also a simplex of this standard triangulation, thus it is an integer polytope. This completes the proof of Frank's Sandwich Theorem and thus proves point $a)$.

We obtain assertion $b)$ by remarking that the set of maxima of a supermodular function is a sublattice.

For what concerns point $c)$, then $\mathbf{P}$ is the intersection of two superdifferentials (of the functions U and $-C$) at any optimal matching μ^* . The intersection of in-between-convex sets is in-between-convex. Thus we only need to show the following lemma.

Lemma *The superdifferential of a supermodular function is in-between-convex.*

Proof Let f be a supermodular function and $p' \leq p''$ be two supergradients to f at a point x . We have to check that, for any p such that $p' \leq p \leq p''$, there holds

$$f(y) - f(x) \leq p(y) - p(x). \tag{3}$$

Since p' and p'' are subgradients, there holds

$$f(y \vee x) - f(x) \leq p'(y \vee x) - p'(x)$$

and

$$f(y \wedge x) - f(x) \leq p''(y \wedge x) - p''(x).$$

Now, $y \vee x \geq x$ and $p \geq p'$ imply $p'(y \vee x) - p'(x) \leq p(y \vee x) - p(x)$, and $y \wedge x \leq x$ and $p \leq p''$ imply $p''(y \wedge x) - p''(x) \leq p(y \wedge x) - p(x)$. Thus, we have

$$f(y \vee x) - f(x) + f(y \wedge x) - f(x) \leq p(y \vee x) - p(x) + p(y \wedge x) - p(x). \tag{4}$$

Since p is a modular function $p(y \vee x)+p(y \wedge x) = p(y)+p(x)$ holds, thus (4) can be rewritten as follows

$$f(y \vee x) + f(y \wedge x) - 2f(x) \leq p(y) - p(x). \tag{5}$$

From (5) and from supermodularity of f, i.e. $f(y) + f(x) \leq f(y \vee x) + f(y \wedge x)$, we obtain

$$f(y) - f(x) = f(y) + f(x) - 2f(x) \leq f(y \vee x) + f(y \wedge x) - 2f(x) \leq p(y) - p(x).$$

Therefore, (3) is verified. This completes the proof of the lemma and hence of point $c)$. ∎

In contrast to PM-concave functions, supermodular functions have already a respectable tradition in economics. They appear in the context of convex cooperative games (a concept due to Shapley (1971)) as well as in a number of inventory problems (see Topkis (1978)).

We give here a few examples of sub/supermodular functions and suggest how they can be constructed. The task is slightly simpler than in the case of PM-concave functions, for the set of supermodular functions forms a convex cone. It is not difficult to devise supermodular functions depending on two or three variables; summing such functions yields new supermodular functions.

We can examine another application of this summation principle. Let ν be a non-negative function on 2^{Ω}. Define the function u on 2^{Ω}, where, for $A \subset \Omega$,

$$u(A) = \sum_{B \subset A} \nu(B).$$

u is a supermodular function.

Finally, it is simple to construct anonymous supermodular functions. Let ϕ be a monotone convex function of a single variable. Then the function U on 2^{Ω}, $U(A) = \phi(|A|)$, is supermodular. More: we can substitute the number of elements $|\cdot|$ by an arbitrary positive measure on Ω (see Shapley (1971)).

The functions just defined above are particular instances of the following observation: the composition $\phi \circ U$ of a monotone supermodular function U with a monotone convex function ϕ is supermodular (Topkis (1978), see also Lovász (1983)).

7. Markets with both substitutes and complements

We consider now the following mixed case in which part of the goods on sale are substitutes, whereas the remainder are complements. To this end, we assume that the sellers are divided into two groups $\mathcal{S}_s$ and $\mathcal{S}_c$. The goods supplied by sellers of the first group are mutual substitutes, whereas those supplied by the sellers of the second group are mutual complements. We impose the following three conditions on both utility and cost functions:

Assumption (The Compatibility Principle)

($\mathcal{S}_s$) *the function* $-c(s,\cdot)$ *is* PM*-concave for any* $s \in \mathcal{S}_s$ *;*

($\mathcal{S}_c$) *the function* $-c(s,\cdot)$ *is supermodular for any* $s \in \mathcal{S}_c$ *;*

($\mathcal{B}$) *the utility function* $u(\cdot,b)$ *of any buyer* b *is a sum of a* PM*-concave function* $u_s(\cdot,b)$ *of variables from the set* $\mathcal{S}_s$ *and a supermodular function* $u_c(\cdot,b)$ *of variables from the set* $\mathcal{S}_c$ *.*

Theorem 6 *A two-sided market satisfying the Compatibility Principle has equilibria.*

Proof The rationale of the proof is again the same as in the preceding cases. The aggregate utility function U is the sum of two functions U_s and U_c, where the first is a *PM*-concave function of the variables in the set $\mathcal{S}_s \times \mathcal{B}$, while the second is a supermodular function of the variables in the set $\mathcal{S}_c \times \mathcal{B}$. The cells of U are Cartesian products of two types of polytopes : integer polymatroids in the space $\mathbb{R}^{\mathcal{S}_s \times \mathcal{B}}$ and simplexes of the standard triangulation of the unit cube in the space $\mathbb{R}^{\mathcal{S}_c \times \mathcal{B}}$.

The aggregate cost function C is similarly the sum of a *PM*-convex function of the variables in the set $\mathcal{S}_s \times \mathcal{B}$ and of a submodular function of the variables in the set $\mathcal{S}_c \times \mathcal{B}$. There, as well, the cells of $-C$ are Cartesian products of two types of polytopes : integer polymatroids in the space $\mathbb{R}^{\mathcal{S}_s \times \mathcal{B}}$ and simplexes of the standard triangulation of the unit cube in the space $\mathbb{R}^{\mathcal{S}_c \times \mathcal{B}}$.

The intersection of two such cells is an integer polytope. Therefore, the Proposition 3 obtains for the functions U and C, whence existence of equilibria. ∎

One readily sees that the set $\mathbf{M}$ of equilibrium matchings is the cartesian product of an in-between-convex subset of $2^{\mathcal{S}_s \times \mathcal{B}}$ and of a sublattice in $2^{\mathcal{S}_c \times \mathcal{B}}$. The set $\mathbf{P}$ of equilibrium prices is the cartesian product of a sublattice of $\mathbb{R}^{\mathcal{S}_s \times \mathcal{B}}$ and of an in-between-convex subset of $\mathbb{R}^{\mathcal{S}_c \times \mathcal{B}}$.

The same results hold true in the more general context, in which the set Ω of all goods is partionned into two groups - a group of substitutes and a group of complements. In this case, we also need to make use of a modified Compatibility Principle, which imposes that if two goods are substitutes with respect to consumption purposes, then they should be so with respect to production purposes, and identically for complements. Nevertheless, we have no satisfactory explanation about why the set of goods could turn out to be partionned in such a way. Last, this principle of compatibility justifies the recourse to a gross substitution argument in Kelso and Crawford's one-to-many job matching model. Indeed. Since workers are allowed to work for a single firm only, their cost functions are of the *GS*-type (see Example 4). The principle of compatibility requires that utility functions of firms (actually their gross product functions) be of the *GS*-type as well. Similarly, when buyers need at most one item, then their utility functions are of the *GS*-type (see Example 5). In contrast, observe that the Compatibility Principle is in fact (and not surprisingly so) violated in Examples 1 and 3, where we fail to exhibit a competitive equilibrium.

References

Bevia, C., M. Quinzii, and J. Silva (1999). Buying several indivisible goods. *Mathematical Social Sciences 37*, 1–23.

Crawford, V. P. and E. Knoer (1981). Job matching with heterogeneous firms and workers. *Econometrica 49*, 437–450.

Danilov, V. and G. Koshevoy (2000). Discrete convexity and unimodularity i. Technical report, Central Institute of Economics and Mathematics.

Danilov, V., G. A. Koshevoy, and A. Sotskov (1994). Equilibrium in a market of intellectual goods. *Mathematical Social Sciences 27*, 133–144.

Danilov, V. I., G. Koshevoy, and K. Murota (1998). Equilibria in economies with indivisible goods and money. Mimeo, Research Institute for Mathematical Sciences, Kyoto University, Kyoto Japan.

Danilov, V. I., G. A. Koshevoy, and C. Lang (2001). Gross substitution, discrete convexity and submodularity. January 2001. submitted to the Journal of Discrete Applied Mathematics, CEMI, Russian Academy of Sciences, Moscow.

Danilov, V. I. and C. Lang (2000). The gross substitution property in piece-wise affine economies (svojstvo valovoj zamenimosti dlja kusochno-linejnyx ekonomik). December 2000. submitted to Mathematicheskie i Ekonomicheskie Metodi, CEMI, Russian Academy of Sciences, Moscow.

Frank, A. (1982). An algorithm for submodular functions on graphs. *Annals of Discrete Mathematics 16*, 97–120.

Frank, A. and E. Tardös (1988). Generalized polymatroids and submodular flows. *Mathematical Programming 42*, 489–563.

Gale, D. and L. Shapley (1962). College admissions and the stability of marriage. *American Mathematical Monthly 69*, 9–15.

Gul, F. and E. Stacchetti (1999). Walrasian equilibrium with gross substitutes. *Journal of Economic Theory 87*, 95–124.

Kaneko, M. (1982). The central assignment game and the assignment markets. *Journal of Mathematical Economics 10*, 1483–1504.

Kelso, A. and V. Crawford (1982). Job matching, coalition formation and gross substitutes. *Econometrica 50*, 1483–1504.

Koopmans, T. and M. J. Beckmann (1957). Assignment problems and the location of economic activities. *Econometrica 25*, 53–76.

Lovász, L. (1983). Submodular functions and convexity. In M. G. A.Bachem and B. Korte (Eds.), *Mathematical Programming: The State of Art*, pp. 235–257. Springer- Verlag.

Murota, K. (1996). Convexity and Steinitz's exchange property. *Advances in Mathematics 124*, 272–311.

Polterovich, V. M. and V. Spivak (1982). Mappings with gross substitution in the theory of economic equilibrium (Otobrazhenja s valovoj zamenimost'ju v teorii ekonomicheskogo ravnovesija). In *Current Problems in Mathematics*, pp. 111–154. Vsesojuznij Institut Nauchnoj i Tekhnicheskoj Informatsii.

Quinzii, M. (1984). Core and competitive equilibria with indivisibilities. *International Journal of Game Theory 13*, 41–60.

Roth, A. (1984). Stability and polarization of interests in job matching. *Econometrica 52*, 47–57.

Samuelson, P. (1947). *Foundations of Economic Analysis.* N.Y.: Atheneum.

Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory 1*, 11–26.

Shapley, L. S. and M. Shubik (1972). The assignment game I: the core. *International Journal of Game Theory 1*, 111–130.

Topkis, D. M. (1978). Minimizing a submodular function on a lattice. *Operations Research 26*, 305–321.