Efficiency and Information Aggregation in Auctions

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First Draft: December 1995

This Draft: August 1998

Abstract

There is an underlying tension between allocative efficiency and information aggregation in markets. We explore this in the context of an auction in which k identical objects of unknown quality are auctioned off to n bidders. Bidders receive a signal that gives some information about the quality of the objects, and in addition, differ in their taste for an object of any given quality. The k highest bidders get an object and pay a price equal to the $k+1$st highest bid. We find conditions under which in the limit, objects are allocated efficiently to those with the highest tastes, and price converges in probability to the value of an object to the marginal taste type.

1. Introduction

In many market settings, there is both a private and a common component to valuations. In this paper, we consider situations where market participants with idiosyncratic tastes have private information about the quality of the goods for sale. As an example, consider the market for new cars. Potential buyers may have private information about the reliability of a particular model and also differ in their valuations of the styling and the features of the car. An example in an auction setting is the sale of timber harvesting contracts on public forests: firms differ in their harvesting costs and, in addition, are asymmetrically informed about the quality of the tracts from a particular forest.[1]

To capture this, we consider a setting with a fixed supply of identical objects of unknown quality q. Each agent has an idiosyncratic taste parameter t_i and places value $u_i(q, t_i)$ on winning a single object (and no value on further objects). An example that satisfies our assumptions is

$$u_i(q, t_i) = q + t_i.$$

Each agent knows his own taste parameter, t_i, and receives a noisy signal about quality. In the simplest example, the agent receives one of two signals where the probability of receiving signal 1 is strictly decreasing in q, conditionally independent across players given q, and independent of the t_i's.

A common intuition in the economics literature is that if there are many "small" market participants then the market outcome will be efficient, and the market price will aggregate the information dispersed in the economy. However, this intuition is not complete.

Assume in the context of our example that there is a continuum of buyers with mass 1 and that the various t_i's are uniformly distributed on $[0, 1]$ and independent of all other variables. Assume further, there is a continuum of objects with mass 1/2. Since there is a continuum of buyers with independent signals about q, knowledge of the fraction of agents with signal 1 reveals q. So, dispersed among market participants

[1] See Haile (1996) for a description of timber auctions.

is full information about q. And, the market has a fully revealing rational expectations equilibrium (REE) given by $p = q + 1/2$. Demand is equal to the fraction of consumers for whom $E[q|p] + t_i > p$. With $p = q + 1/2$ this inequality holds for all consumers with $t_i > 1/2$ and hence demand (and supply) is 1/2. Since buyers can infer the true quality of the object from the price, their demands are independent of their private information. Price thus aggregates private information by assumption, and allocative efficiency follows automatically since demands then depend only on tastes.

However, as in all competitive models, there is no explanation given in the REE model of how this price came about. Since agents' demands depend only on t_i, how did the price come to incorporate knowledge of quality?[2] To fully understand this market, we must consider a model where price is a function of individual buyer behavior, and where this behavior in turn depends only on individuals' private information (and not on the information contained in the equilibrium price). For example, consider an auction implementation of this market. Each bidder chooses a bid $b(t_i, 1)$ or $b(t_i, 2)$ depending on his taste parameter and the signal he observes. Players whose bid is above the median receive an object at price equal to the median bid.

Suppose that bidders use their private information about quality and hence that $b(t_i, 2) > b(t_i, 1)$. Since the fraction of bidders who receive a good signal is increasing in q this implies that the median bid is a strictly increasing function of q. Now observe that a small change in a bid only affects the payoff of the bidder if it is on the margin between winning and losing the object, i.e., if the bid is equal to the median bid. But since the median bid is strictly increasing in q, the bidder can infer the precise quality of the objects conditional on this event. This implies that the optimal bid only depends on the bidder's private valuation t_i and is independent of his signal about quality.

Thus, it appears that the REE described above cannot be implemented by a straightforward non-cooperative bidding model. The reason is that agents would like to bid the same amount, independent of their private information, when other agents bid informatively.

On the other hand, if all other bidders ignore their private information, i.e., $b(t_i, 2) = b(t_i, 1)$, then price will be uninformative about quality. But then each

[2] See Milgrom (1981) on this point.

bidder has an incentive to use his private information. Hence we can also exclude the existence of equilibria where information is not aggregated.

These arguments have a flavor similar to the Grossman-Stiglitz paradox (Grossman and Stiglitz, 1976). They point out the following problem in rational expectations models with a small cost of acquiring information: if the equilibrium price reveals information, then there is no value to acquiring information. But, if no information is acquired, then price is uninformative, and then there is an individual incentive to acquire information. In our setting, the paradox remains even if there is no cost to acquiring information: if the price is very informative about quality, then individuals have no incentive to *use* their information. But, then price cannot be informative. Conversely, if the price is uninformative then individuals have an incentive to use their private information which implies that the price must be informative.

Instead of a continuum population, consider a finite auction with n bidders and k objects. The k highest bidders receive an object and pay the $k+1$st highest bid, with ties broken by symmetric randomizations. For any finite k and n, market participants have some incentive to use their private information even if other market participants do the same. Hence the Grossman-Stiglitz style problem just described does not occur in the finite setting. However, for fixed k and n there is a tension between information aggregation and allocative efficiency: the more sensitive bids are to private information, the more information is aggregated in the price but also the greater is the allocative inefficiency.

The contribution of this paper is to show how as k and n grow, the tension between information aggregation and allocative efficiency can disappear. Our main result can be summarized as follows:

> Consider a sequence of auctions of the type described. If both k and n go to infinity (and k/n remains bounded away from 0 and 1), then in the limit of any sequence of symmetric equilibria there is both allocative efficiency and full information aggregation. That is, in the limit, objects are allocated to the players with the highest t_i's, and price reflects the true value of an object to the marginal taste type.

To provide an intuition, note first that efficiency and information aggregation are both possible in the limit of the finite game: as the market grows each participant's bidding behavior can become less and less sensitive to his private information while aggregate behavior still reveals quality with increasing precision, so that in the limit both information aggregation and efficiency hold. Thus, the limit of the finite game allows for an outcome that the continuum example above misses.

Of course, this is only the statement that there need not be a conflict between information aggregation and efficiency. It says nothing about *equilibrium* behavior. However, the forces which generate a paradox in the continuum model are precisely those which imply that in the limit of our finite model there is both allocative efficiency and full information aggregation. If for large n and k the equilibrium price does not essentially reveal the quality of the objects, then there is a strong incentive for bidders to use their private information about quality. But then price will reveal the quality of the objects, a contradiction. Conversely, because the equilibrium price reveals the quality of the objects with great precision, optimal behavior is almost independent of private information. Therefore, in the limit, the misallocation of goods is negligible.

To analyze the robustness of our results to the information structure, we consider a model with a finite number of signals, where the probability of getting information may depend both on the valuation of the bidder and on the number of market participants. In this way our model can capture situations where for example information is costly and only some players choose to acquire information. We demonstrate that asymptotic efficiency holds irrespective of the information structure. The idea behind this result is that efficiency can obtain either because bidders largely ignore their information (as they will if asymptotically price reveals quality) or because most bidders simply have no private information.

On the other hand, the conditions under which information aggregation occurs are more stringent: our proof holds only if a non-vanishing fraction of bidders is informed and if information is independent of the bidder's type. Not surprisingly, this rules out situations where information is costly.

In a companion paper (Pesendorfer and Swinkels (1997)), we analyze information aggregation in a setting similar to this but in which tastes are homogeneous. There we show that information aggregation holds if and only if $k_r \to \infty$ and $n_r - k_r \to \infty$. The result for the more general setting of this paper is weaker in two ways. First, we require that k_r/n_r remains bounded away from 0 and 1 along the sequence. The result for the case when k_r/n_r heads to a boundary in the pure common value setting depends on a fuller characterization of the equilibrium than we can achieve in this setting. The second weakness is more serious: in the pure common value case, we are able to show that there exists a unique symmetric equilibrium, fully characterize it, and then show that it has the properties needed for information aggregation. In this setting, we are unable to show existence of a symmetric equilibrium, or provide a full description of the equilibrium. Rather, we show that if symmetric and increasing equilibria exist, then they must have the properties necessary for our results. In the final section of the paper we show that $\varepsilon-$equilibria satisfying these conditions exist.

The present paper is also related to work by Feddersen and Pesendorfer (1996). They analyze two candidate elections in which voters have different preferences and have private information about the quality of the candidates. Feddersen and Pesendorfer give conditions under which the election fully aggregates the private information of voters.

The major impetus for our work is to understand the general conflict between information aggregation and allocative efficiency, rather than the limiting behavior of large auctions themselves. Auction models are convenient for this task because they are a tractable non-cooperative model of price setting. However, even for auctions, our results are of some relevance: many real auctions do have a large supply and set of potential buyers. And, as we argue below, auctions which combine a private and common component of values probably should be viewed as more the norm than the exception. The model seems to fit the timber example reasonably well. For general markets it would be desirable to consider strategic sellers as well as strategic buyers. Such a generalization is left for future research.

In the next section, we lay out the model, and describe the key properties of the equilibrium for fixed k and n that we will need for our asymptotic results. We also briefly discuss the role of our two dimensional type space. Section 3 is the heart of the paper. It examines the behavior of the equilibrium as k and n grow large. Finally, our asymptotic efficiency and information aggregation results allow us to characterize asymptotic bidding behavior precisely. This is despite the fact that equilibria are extremely difficult to solve for in the finite setting. Section 4 provides this characterization. Section 5 discusses $\varepsilon-$equilibria, while Section 6 concludes. Proofs of all results are contained in Section 7.

2. The Model and Equilibrium for Fixed Market Size

There are n buyers and k identical objects for sale. The quality of the objects, q, is drawn from the interval $[0, 1]$ according to a distribution F. We assume that F has a continuous and strictly positive density f. Buyer i's taste parameter t_i is drawn independently from the interval $[0, \bar{t}]$ according to the probability distribution W; W has the continuous and strictly positive density w.

Buyer i's utility from a single object is $u(q, t_i)$. Further objects give utility 0. We assume that u is continuously differentiable with both partial derivatives strictly greater than zero. Thus, the valuation is increasing in t_i and q.

Buyer i knows his own taste parameter, t_i, and receives a signal $s_i, 1 \leq s_i \leq S$ about quality. We assume that for a given quality, the signals s_i are independent across players. Therefore we can define $\pi(s|q,t)$ as the probability that type t receives signal s when the quality is q.

We allow the probability of receiving signals to depend on the taste parameter. This captures situations where some taste types have access to better information than others. This may come about because information is costly and different taste types have different incentives to acquire information. For example, suppose there are two signals but not all bidders have access to (or purchase) these signals. Our model can incorporate this by defining an information structure with three signals where one of the three signals represents "no information". Types who do not have access to information receive the uninformative signal with probability one.

We assume that the effects of t and of q on the signals received are *independent.* Thus for example, t can affect whether or not particular information is purchased, but does not affect the outcome of the signal if information is purchased. Formally, this is described by the assumption that π can be written as a product of two functions

$$\pi(s|q,t) = \pi_1(s|q)\pi_2(s|t).$$

In addition, we require that higher signals are associated with higher levels of q. Formally, this is expressed by the monotone likelihood ratio property. Below we summarize our assumptions on π.

Assumption 1. *$\pi(s|q,t)$ satisfies:*

1. *Independence: There are functions $\pi_1(s|q), \pi_2(s|t)$ such that*

$$\pi(s|q,t) = \pi_1(s|q)\pi_2(s|t).$$

   Moreover, $\pi_1(s|q)$ is continuously differentiable in q and $\pi_2(s|t)$ is a measurable function of t.

2. *Limited Information: There is $\eta > 0$ such that $\pi_1(s|q) \geq \eta > 0$ for all s and all q.*

3. *Monotone Likelihood Ratio Property: For all $s' > s$, the ratio $\pi_1(s'|q)/\pi_1(s|q)$ is strictly increasing in q.*

4. *$\sum_s \pi_1(s|q) = 1$ for all q and $\int \pi_2(s|t)w(t)dt > 0$ for all s.*

Part 2 of Assumption 1 implies that signals provide only noisy information about q: A player who begins with full support beliefs will retain them after seeing a signal. Part 4 of Assumption 1 says that π_1 can be interpreted as a probability distribution (this is a normalization) and that each signal is received with strictly positive probability (this is a convenience).

Example 1 gives an information structure that satisfies our assumptions.

Example 1 There are two informative signals: a "good" and a "bad" signal. The higher the quality the more likely it is that the good signal is received. Specifically, the good signal is received with probability q and the bad signal with probability $1-q$. Depending on their taste type some bidders have access to the signals and some do not. We capture this situation by defining the following information structure: there are three signals $s \in \{1, 2, 3\}$ where signals 1 and 3 play the role of the original pair of signals and signal 2 represents "no-information". Let

$$\pi_1(s|q) = \begin{cases} q/2 & \text{if} \quad s=1 \\ 1/2 & \text{if} \quad s=2 \\ (1-q)/2 & \text{if} \quad s=3 \end{cases}$$

and let

$$\pi_2(s|t) = \begin{cases} 2\Pr(t \text{ receives information}) & \text{if} \quad s=1 \\ 2(1-\Pr(t \text{ receives information})) & \text{if} \quad s=2 \\ 2\Pr(t \text{ receives information}) & \text{if} \quad s=3 \end{cases} .$$

Thus if type t receives information with probability 1 then his information reduces to the original pair of signals. On the other hand, if type t receives no information then he receives signal 2 independent of q.

Each bidder i submits a bid b_i as a function of his taste-signal pair (t_i, s_i). The k highest bidders receive an object and pay the $k+1$st highest bid. If two or more bidders are tied at the k-th highest bid then each such bidder has an equal chance of receiving the object. The overall payoff of a winning bidder is $u(t_i, s_i) - b_i$. The payoff of a losing bidder is 0.

We consider symmetric Nash equilibria in pure strategies.[3] Thus we can describe equilibrium behavior by a bidding function $b(t, s)$. The following assumption requires that the bidding behavior captured by $b(t, s)$ is continuous and strictly increasing in t.[4]

[3]The restriction to pure strategies is without loss of generality. To see this note that if for a given signal s, b is optimal for taste t, then every optimal bid with signal s and taste $t' > t$ is at least b. This is so as t contains no information about either q or other players' actions, and so increasing t simply makes the increased probability of winning resulting from the increase in bid strictly more attractive. But then, for almost all t, there is a unique optimal bid. (To see this, let $y(t)$ be a selection from the best response correspondence for some given s. Then, $y(t)$ is a non-decreasing function, and $y(t)$ jumps at every t where the best response is non-unique. So, there are at most a countable set of such points.) Since the distribution of t is atomless, for every equilibrium, there is a realization equivalent equilibrium in pure strategies (simply take a selection at the zero measure set of points where the best response is not unique).

[4]Note that the argument in the previous footnote implies that the bidding behavior is weakly

Assumption 2. *$b(t,s)$ is strictly increasing in t, continuous in t, and differentiable in t at all but* finitely *many points.*[5]

In the following we describe the equilibrium strategies from the perspective of bidder 1. (Since we are working with symmetric equilibria, this is enough.) Let d denote the kth highest bid among all bidders except bidder 1. When $b_1 = d$, then a small increase in 1's bid implies that he wins an object whereas a small decrease in his bid implies that he does not win an object. We thus refer to the event $d = b$ as "b is pivotal".

Our first result says that bids are increasing in the signal received. Moreover, each buyer bids exactly what the object is worth to him if his bid is pivotal.

Proposition 1. *If Assumption 2 holds, then in equilibrium $b(t,s') > b(t,s)$ for all $s' > s$ and all t. Moreover,*

$$b(t,s) = E\left(u\left(q,t\right)|d = b(t,s), s\right) \tag{2.1}$$

for all s and t.

The monotonicity of the bidding behavior in the signal received is a consequence of the monotone likelihood ratio property and the continuity of the bidding functions.

To give an intuition for Equation (2.1), consider a small increase in player 1's bid, say from b to $b+\varepsilon$. If $d < b$ then this change is irrelevant since 1 wins with either b or $b+\varepsilon$ and pays the same amount, d, in either case. If $d < b+\varepsilon$ the change is again irrelevant since 1 loses with either bid. The only situation in which the change

increasing in t. Assumption 2 strengthens this to strict monotonicity and continuity.

In a previous version of this paper (Pesendorfer and Swinkels 1996), we assumed only two signals. In that case the assumption of continuity of the bidding functions is unnecessary. At first blush, it seems obvious that bidding functions must be strictly increasing in t, since in the presence of a tie, either high t types should want to bid a little more, or low t types a little less. The difficulty comes in the fact that when there is a tie at b, winning or losing may contain extra information about q. We have thus been unable to rule out that the possibility of equilbria involving flat spots in the bidding functions, supported by a form of upside down winner's curse at the associated bid (that is, the news that one wins with a bid of b contains positive information conditional on the event that the pivotal bid is at b, and losing contains negative news). See Pesendorfer and Swinkels (1996) for a full discussion.

[5]Observe that since $b(t,s)$ is monotone it is automatically differentiable almost everywhere. The assumption that differentiability fails at most at finitely many points is needed for technical reasons.

matters is thus when $d \leq b \leq d + \varepsilon$, in which case bidding $b + \varepsilon$ instead of b wins an object at a price of approximately b. Player 1's expected utility from the object in this event is

$$E\left(u\left(q,t\right)\left|b \leq d \leq b+\varepsilon, s\right.\right).$$

Taking ε to zero and noting that at the optimal bid, $b(t, s)$, small changes in bid should leave payoffs unaffected, yields Equation 1.

2.1. Comments on the Two Dimensional Type Space

Our model has the property that each bidder is characterized by two parameters: his taste and his signal about quality. It is this two-dimensionality of the type space that gives rise to the conflict between information aggregation and allocative efficiency discussed in the introduction.

In contrast to our model, the vast majority of the existing auction literature works with a one-dimensional type space: each player receives a single real valued signal. Milgrom and Weber (1982) show that despite this one dimensional type space, the model is rich enough to include pure private values (a player's utility from the object depends only on his own signal), pure common values (a player's utility from the object depends symmetrically on all signals) and some models which are intermediate between private and common values (a player's utility depends on all signals, but weighs signals of other players differently than his own).

In the one-dimensional environment there can be no-conflict between information aggregation and efficiency: a higher estimate of quality always also implies a higher taste parameter. We are thus unable to explore this conflict using the standard Milgrom Weber (1982) framework. Moreover, we would argue that most auction settings contain both common and private values components. And, unless tastes and information are perfectly correlated, this requires a type space with more than one dimension.[6] For example, bidders on an oil lease may differ in their current cost of exploration and drilling activity, and in addition, have different information about the amount of oil which a certain tract might contain. It seems highly artificial to

[6]More correctly, while one can always map a multidimensional type space into a single dimensional type space, doing so while maintaining any sort of monotonicity is impossible.

assume that good information about the oil-content of a tract also implies that the company has lower costs of exploration and drilling.

With a single dimensional type space, there is a natural guess about the form of the equilibrium: bids will be strictly increasing in type (and symmetric across players). Having made this guess, inference problems about equilibrium behavior are reduced to inference problems about underlying parameters. So for example, the question "what would I infer if I knew my *bid* was tied with the highest *bid* by my opponents" reduces to "what would I infer if I knew my *signal* was tied with the highest *signal by my opponents*." One can then easily derive first order conditions on what the bid of any given signal type must look like. Integrating these first order conditions yields a candidate equilibrium, and the assumption of affiliation allows one to verify that the candidate is indeed an equilibrium.

With a two dimensional type space, there is no "natural" complete ordering on the type space. Guessing an order on the type space involves guessing which pairs of signals and tastes go together: e.g., for any given type t, signal s and signal s' what type t' has the property that $b(t', s') = b(t, s)$? But this question cannot be answered independent of the equilibrium strategies. It is equivalent to guessing what a bidder infers from $d = b$. This in turn of course depends on how sensitive the bids of other players are to their information.

As a consequence of the two-dimensional type space we have not been able to prove existence of equilibria (although see Section 5 on ε-equilibria). Our results instead hinge on a partial characterization of what equilibria must look like in the limit, if they exist.

3. Large Auctions

We now turn to the questions of asymptotic efficiency and information aggregation in large auctions. To do so, we characterize the limit of auction outcomes as the number of buyers and the number of objects converge to infinity. Formally, we consider a sequence of auctions indexed by r. Auction r has k_r objects and n_r bidders, where both k_r and n_r go to infinity as $r \to \infty$. We assume that the ratio of objects to bidders, k_r/n_r, stays away from zero and one along the sequence.[7] Thus, a situation where there are always half as many objects as buyers satisfies our assumptions, but one where the number of objects stays fixed as the number of buyers increases does not.

Assumption 3. $n_r \to \infty$ *as* $r \to \infty$*. Moreover, there is a* $\beta > 0$ *such that* $1 - \beta > k_r/n_r > \beta$ *for all* r*.*

To indicate that we are working with elements of a sequence we will subscript all relevant objects such as strategies and inverse bidding functions by r.

Along this sequence of auctions we keep the prior distribution of quality and the distribution of taste types fixed. As before, we allow the probability of receiving an informative signal to vary with type. However, in addition we wish to allow the information structure $\pi(s|q,t)$ to vary with r. The motivation for this is that in some auction settings, information acquisition will be endogenous. And, for example, it may be that in large auctions, a smaller fraction of players choose to acquire information than in auctions with a smaller number of players. To capture this, we assume that the information structure takes the form

$$\pi_r(s|q,t) = \pi_1(s|q)\pi_{2r}(s|t). \tag{3.1}$$

Hence we assume that π_1 stays fixed along the sequence while π_2 may change with r. Thus, the information a signal reveals about the quality of the objects stays fixed, but the probability that a particular taste type receives a particular signal may vary in r. The assumption is illustrated by the two examples which follow.

Example 2 There are two informative signals. Each bidder receives information with probability $1/r$. Hence, as r increases, the fraction of bidders who are informed

[7] In a companion paper (Pesendorfer and Swinkels (1997) dealing with a pure common values setting, we were able to explicitly characterize the equilibrium, and this allowed us to derive an information aggregation result even when k_r/n_r went to 0 or 1. Here, we are only able to derive some properties of the equilibrium: with this weaker characterization we require the stronger assumption on k_r/n_r to obtain results. Whether our results would go through when k_r/n_r does go to 0 or 1 is an open question.

decreases to zero. An example of such an information structure is given by

$$\pi_1(s|q) = \begin{cases} q/2 & \text{if} \quad s = 1 \\ 1/2 & \text{if} \quad s = 2 \\ (1-q)/2 & \text{if} \quad s = 3 \end{cases}$$

and

$$\pi_{2r}(s|t) = \begin{cases} 2/r & \text{if} \quad s = 1 \\ 2(1-1/r) & \text{if} \quad s = 2 \\ 2/r & \text{if} \quad s = 3 \end{cases}$$

Signals s_1 and s_2 are informative about q, s_2 is not. In this example, the probability of receiving an informative signal is the same for each bidder; it varies only with r.

Example 3 We may also extend Example 1 to this setting. Bidders first learn their taste parameter t_i and then have to decide whether to purchase information or not. The decision to purchase with any given t_i may depend on the number of market participants. If information is purchased, it has the same structure as s_1 and s_3 in Example 2. Any symmetric information purchase behavior in this setting gives rise to an information structure that can be captured by the function π_1 above and by the following function π_{2r} :

$$\pi_{2r}(s|t) = \begin{cases} 2(\Pr(\text{type } t \text{ acquires information}|r) & \text{if} \quad s = 1 \\ 2(1 - \Pr(\text{type } t \text{ acquires information}|r)) & \text{if} \quad s = 2 \\ 2(\Pr(\text{type } t \text{ acquires information}|r) & \text{if} \quad s = 3 \end{cases} \cdot$$

The functions $\pi_1$and π_{2r} for each r are assumed to satisfy A1. Note in particular however, that while part 4 of Assumption 1 requires that every signal is received with positive probability for each r, it does not place any lower bound on the probability with which a signal is received. Thus, in Example 2, the probability of receiving signal 1 or 3 is positive for each r, but tends to 0 as $r \to \infty$.

3.1. Asymptotic Efficiency

Efficiency requires that the players with the k highest values are those who win objects. This maximizes the gains from trade across buyers and the seller, and, given our assumption of quasi-linear utility functions, is the unique Pareto optimal allocation. Of course, given that a player who observes $s' > s$ always bids more than one who observes s, exact efficiency is unattainable for any finite auction. In particular, there is a positive probability that a bidder with type (t, s') wins the object while a bidder with type (t', s), $t' > t$ does not.

For given r, let the random variable $G_r^*(q)$ denote the maximal gains from trade, and let the random variable $G_r^e(q)$ denote the equilibrium gains from trade.[8] Let

$$L_r(q) \equiv \frac{G_r^*(q) - G_r^e(q)}{n_r}$$

be the per capita difference between the maximal and realized gains to trade for given q.[9] We use $L_r(q)$ as our measure of inefficiency:

Definition 1. *A sequence of equilibria is* asymptotically efficient *if uniformly for all* q

$$L_r(q) \to 0$$

in probability as $r \to \infty$.

We now turn to the question of when asymptotic efficiency holds.

Proposition 2. *Under Assumptions 1-3, any sequence of equilibria is asymptotically efficient.*

[8] $G_r^*(q)$ and $G_r^e(q)$ are random variables since they depend on the particular realization of taste parameters $(t_1, ..., t_{n_r})$ and signals $(s_1, ..., s_{n_r})$.

[9] Because k_r/n_r is bounded from 0 and 1, it makes no difference if we measure losses on a per unit or per person basis.

To give an intuition for Proposition 2 it is convenient to first define the inverse of the bidding function. For each signal s, define $t_s(b)$ so that $b(s, t_s(b)) = b$ for every bid made with signal s. This is illustrated by Fig. 1. Since $b(s, t)$ is a strictly increasing and continuous function, $t_s(b)$ is well defined and unique.

Fig. 1. The inverse bidding function

The idea of the proposition can be summarized as follows: inefficiency occurs only if the inverse bidding functions stay apart at a particular bid b (see Fig. 1). But this in turn implies that the event that b is pivotal is very informative about quality. But then, the signal cannot significantly change a bidders' estimate of quality, contradicting that the inverse bidding functions stay apart.

To be more precise, assume that there are two signals, $s = 1, 2$ and that (contrary to the proposition) $L_r(q) > \varepsilon$ for some q. This implies that for some bid b, the taste type who bids b with signal 1 is strictly greater than the taste type who bids b with signal 2. Formally, $t_{1r}(b) - t_{2r}(b) > \varepsilon$. Recalling that W, the distribution of taste types, has density bounded away from 0, this implies that $W(t_{1r}(b)) - W(t_{2r}(b)) > \varepsilon'$. That is, the fraction of taste types who bid below b with signal 1 is strictly larger than the fraction of taste types who bid below b with signal 2, even in the limit.

Assume, for simplicity, that b is in the range of bids made with each signal. Then Proposition 1 implies that

$$E[u(q, t_{1r}(b))|d_r = b, 1] = E[u(q, t_{2r}(b))|d_r = b, 2] = b \tag{3.2}$$

But since $t_{1r}(b) - t_{2r}(b) > \varepsilon$ this equation can only hold if the inference about quality made by a bidder with a good signal when conditioning on the event $d_r = b$ is significantly different from the inference made by a bidder with a bad signal. Formally, it must be the case that for all r and some $\varepsilon'' > 0$

$$E[q|d_r = b, 2] - E[q|d_r = b, 1] > \varepsilon''. \tag{3.3}$$

We will argue that (3.3) cannot be satisfied for large r.

To see this, first observe that the probability that a bidder bids below b is strictly decreasing in q. This follows since the probability that a bidder bids below b is larger (by at least ε') if he receives signal 1 than if he receives signal 2. With higher q fewer bidders receive signal 1 and hence the probability that any given bidder bids below b decreases. Now, if $d_r = b$ then exactly k_r of the $n_r - 1$ bids by players $2, \ldots, n_r$ are above b. If r is large, then the fraction of bids above b is unlikely to be much different from its expectation. Hence, the bidder's estimate of quality conditional on $d_r = b$ must be very tightly distributed around the unique q that satisfies

$$(1 - W(t_{1r}(b))) \cdot \pi(1|q) + (1 - W(t_{2r}(b))) \cdot \pi(2|q) = \frac{k_r}{n_r - 1}.$$

That is, around the q for which the expected number of bids above b is $\frac{k_r}{n_r-1}$. But the fact that the estimate is tightly distributed, along with the fact that individual signals contain limited information (part 2 of Assumption 1) implies that the private signal s cannot significantly change the bidder's estimate of the quality of the object significantly once he conditions on the event $d_r = b$. But then (3.3) cannot be satisfied.

This provides intuition for the result when the probability of an informative signal remains constant. How about a situation like Example 3, where the probability that bidders receive an informative signal (signals 1 and 3) goes to zero? The key is that if in the limit almost everyone receives the same signal (signal 2), then efficiency is automatic since $b(2,t)$ is increasing in t. For inefficiency, it must be the case both that $t_{1r}(b)-t_{3r}(b)$ remains large, and that a non-vanishing fraction of bidders chooses to become informed. But, then we are back in the setting of our previous discussion, since the fraction of people who bid above b will be strictly increasing in q, and hence $d_r = b$ will be very informative about q, contradicting that $t_{1r}(b) - t_{3r}(b)$ stays large.

3.2. Information Aggregation

As a benchmark for information aggregation we use the "full information" market, i.e., the environment where all buyers know the true quality q of the object . Clearly, this is the relevant benchmark only in the case where bidders could infer the true quality with high precision if they observed all the signals. So, for this section (and indeed for the balance of the paper), we will assume that the signal structure stays constant as r varies. In addition, we simplify the information structure so that the probability of receiving a particular signal depends only on q. Thus, we assume

$$\pi_r(s|q,t) = \pi(s|q)$$

for all r.

This assumption could be weakened substantially. See, for example, Pesendorfer and Swinkels (1997, Section 3.5), where information aggregation obtains as long as the fraction of informed bidders does not go to zero too fast. However, it clearly cannot be relaxed entirely. If, for example, the number of people who receive information remains finite, then even if price aggregated all available information, our full information benchmark would fail to be satisfied. Similarly, in the case of costly signals, it must be the case that the price does not fully reveal q in the limit for there to be an incentive for players to purchase information.

With full information, the bidding behavior in any symmetric equilibrium is for each bidder simply to bid his valuation $u(t_i, q)$. Thus the equilibrium price will be equal to the $k_r + 1$st highest valuation. Define $t_r^* = W^{-1}(\frac{n_r - k_r - 1}{n_r})$ so that in expectation, a fraction $(k_r + 1)/n_r$ of bidders have t above t_r^*. The law of large numbers (plus the fact that $u(q,t)$ is continuous in t) then implies that the equilibrium price of the full information game converges in probability to $u(q,t_r^*)$ as r grows large.

We demonstrate in our next proposition that the equilibrium price of the market where quality is unknown also converges to $u(q,t_r^*)$ in probability. Thus, the equilibrium price in a large market is very close (with high probability) to what it would be if the quality of the object was known to all the buyers.

Let the random variable p_r denote the equilibrium price in auction r. We can then state:

Proposition 3. *Suppose Assumptions 1-3 hold and the information structure* $\pi(.|.)$ *is fixed. Then the equilibrium price,* p_r*, converges to* $u(q,t_r^*)$ *in probability.*

Together Propositions 2 and 3 imply that the equilibrium price in a large market is equal to the valuation of an object to the marginal bidder. Thus, in the limit any bidder who does not buy an object has a valuation less than the equilibrium price and conversely, every bidder who gets the object has a valuation larger than the equilibrium price. This implies that no bidder "regrets" his bid, i.e., no bidder would want to change his bid once the equilibrium price is announced.

To provide an intuition for Proposition 3 consider again the example with two signals. Suppose some bid b is made by type $(t_{1r}(b), 1)$ and by type $(t_{2r}(b), 2)$. Furthermore, suppose that $t_{1r}(b)$ converges to $\hat{t}$. By Proposition 2 we know that it must also then be that $t_{2r}(b)$ converges to $\hat{t}$. By Proposition 1 we know that

$$E[u(t_{1r}(b), q)|d_r = b, 1] = E[u(t_{2r}(b), q)|d_r = b, 2] \tag{3.4}$$

and hence it follows that

$$E[u(\hat{t}, q)|d_r = b, 1] - E[u(\hat{t}, q)|d_r = b, 2] \to 0. \tag{3.5}$$

The last expression implies that in the limit the signal does not affect the expected utility of a bidder once he conditions on $b = d_r$. But recall that by strict MLRP, the probability of receiving the good signal, $\pi(2|q)$, is strictly increasing in q (and

$\pi(1|q)$ is strictly decreasing in q). Hence the only time the signal does not affect a bidder's beliefs about quality is if he can already predict the quality extremely precisely without observing his signal. Therefore, (3.5) implies that the probability distribution over q conditional on $d_r = b$ becomes arbitrarily concentrated around the true value $\hat{q}$.

So, the event that b is pivotal reveals that quality is near $\hat{q}$ with probability close to one. From Proposition 1, it must then be that most of the time,

$$u(\hat{q}, \hat{t}) \cong b$$

when b is pivotal, where $\hat{t}$ is the taste type of the marginal bidder. Now we argue that the same conclusion can be obtained in the event that b is the equilibrium price. To see this, note that the event that b is pivotal occurs when $k_r - 1$ of $n_r - 2$ bidders bid above b and one bidder bids b, whereas the event that b is the equilibrium price occurs when k_r of $n_r - 1$ bidders bid above b and one bidder bids b. So, these two events differ only in the behavior of one player. Since by part 2 of Assumption 1, the behavior of player 1 contains only finite information about q, these two events are roughly equivalent. Hence it follows that when the equilibrium price is b then again the true quality $\hat{q}$ and the taste parameter of the marginal bidder $\hat{t}$ will almost always satisfy

$$u(\hat{q}, \hat{t}) \cong p = b.$$

To complete the argument observe that by Proposition 2 and the law of large numbers the marginal bidder has a type close to t_r^* with probability close to one for large r. Therefore the equilibrium price must be close to $u(t_r^*, q)$ with probability close to one for large r.

4. The Limiting Bid Distribution

In this section we characterize the bidding behavior in the limit as $r \to \infty$ for the case of a constant information structure. For simplicity, we assume in this section that the ratio of objects to bidders, k_r/n_r, converges to a constant $\kappa \in (0, 1)$. Let

$t^* = W^{-1}(1-\kappa)$ be the taste parameter such that the probability of drawing a taste above t^* is equal to the fraction of objects, κ.

The bidding behavior in a large auction takes on a very simple form. Bidders with tastes below $t^* - \varepsilon$ bid as if the true q of the object were 0 whereas bidders with tastes slightly above $t^* + \varepsilon$ bid as if the true q of the object were 1. Bidders with these tastes thus essentially ignore their information. Bidders who lie in a very narrow range around t^* behave in a way that depends sensitively on their information.

Proposition 4. *For all $\varepsilon > 0$, there exists $\bar{r}$ such that for all $r > \bar{r}$,*

(1) for all $t < t^* - \varepsilon$, $u(0,t) \leq b_r(t,s) \leq u(0,t) + \varepsilon$,

(2) for all $t > t^* + \varepsilon$, $u(1,t) - \varepsilon \leq b_r(t,s) \leq u(1,t)$.

Figure 2 illustrates Proposition 4 for the case in which $u(q,t) = q + t$, $\kappa = 1/2$, and the median of W is 1/2.

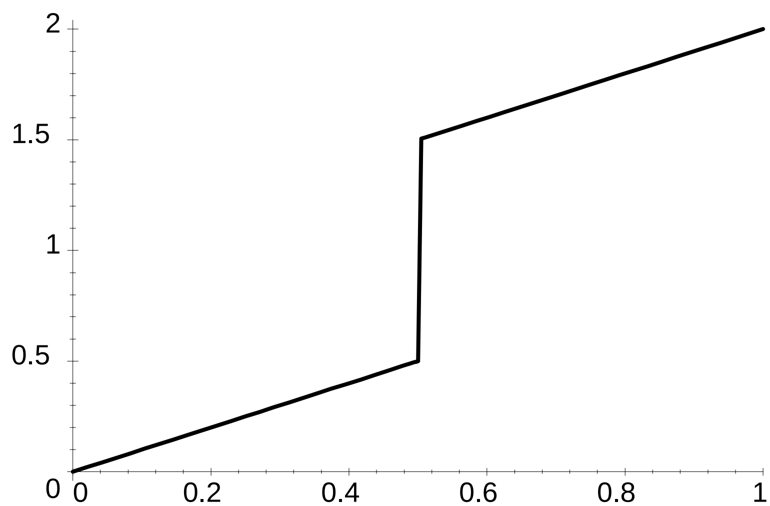

Fig. 2. The limit bid distribution

To give an intuition for Proposition 4 note that since $b(t,s)$ is strictly increasing in t a bidder with $t < t^* - \varepsilon$ expects the equilibrium price (and the pivotal bid) to be larger than his own bid with probability close to one for all values q. This is the case since (by Proposition 3) the fraction of bidders bidding above t is almost always strictly larger than k_r/n_r for large r. If the unlikely event occurs that t is the marginal bidder then it must be the case that both an unusual distribution of t's and a very low value of q have been drawn. A similar argument applies for $t > t^* + \varepsilon$.

This characterization of the equilibrium also makes clearer how the market manages both information aggregation and allocative efficiency. To achieve allocative efficiency, it is enough that the interval of tastes over which bids depend sensitively on information grows narrow. Then, a growing fraction of bidders lie outside of this interval, and thus act essentially independently of their information. On the other hand, while the fraction of bidders who use their information a great deal converges to 0, their absolute number grows fast enough that price contains all information about quality in the limit.

The limiting characterization of the equilibrium has an interesting parallel to Feddersen and Pesendorfer (1996). In that paper, voters have different tastes about two candidates and different information about the candidates. Feddersen and Pesendorfer show that as the number of voters grows large, voters who have tastes either a little to the left of center vote for the left candidate regardless of their information, while voters a little to the right of center vote for the right candidate regardless of their information. Only a narrow band of moderates uses their information in their behavior. The analysis in that paper is simplified by the binary action space (vote left or vote right), which essentially forces players to either use their information a great deal or not at all. Interestingly, our limiting equilibrium approaches that result even though players have available to them strategies which use information to intermediate degrees.

5. Epsilon Equilibria

While we are unable to prove existence of equilibrium we can demonstrate the existence of $\varepsilon-$equilibria in the auction for large n. More precisely, for every ε there is an n such that for $n' > n$ there are continuous and strictly increasing bidding functions such that each type cannot increase his payoff by more than ε. These $\varepsilon-$equilibria satisfy information aggregation and are $\varepsilon-$asymptotically efficient, i.e., the per-capita gains from trade are within ε of their maximum for $n' > n$.

Rather than prove this result for the general class of utility functions and information environments above, we specialize to our simple two signal example to illustrate the point. It should be clear, however, that the argument can be generalized to any constant signal structure (satisfying our assumptions) at the expense of more complicated notation.

Example 4 We assume that there are two signals, where $\pi(1|q) = 1 - q$ and $\pi(2|q) = q$. We assume $k_r/n_r = \frac{1}{2}$ and that t is uniform on $[0,1]$. The payoff function is assumed to be $u(t,q) = t + q$.

The following pair of bidding functions constitute a continuous $\varepsilon-$equilibrium that is asymptotically $\varepsilon-$efficient and aggregates information.

$$b^{\varepsilon}(t,1) = \begin{cases} t & \text{if } \ t \leq 1/2 \\ t + \frac{t-1/2}{\varepsilon} & \text{if } \ 1/2 \leq t \leq 1/2 + \varepsilon \\ t+1 & \text{if } \ t \geq 1/2 + \varepsilon \end{cases}$$

$$b^{\varepsilon}(t,2) = \begin{cases} t & \text{if } \ t \leq 1/2 - \varepsilon \\ t + \frac{t-1/2+\varepsilon}{\varepsilon} & \text{if } \ 1/2 - \varepsilon \leq t \leq 1/2 \\ t+1 & \text{if } \ t \geq 1/2. \end{cases}$$

Fig. 3 depicts the two bidding functions for $\varepsilon = .1$. The upper line depicts $b^{\varepsilon}(t,2)$, and the lower $b^{\varepsilon}(t,1)$.

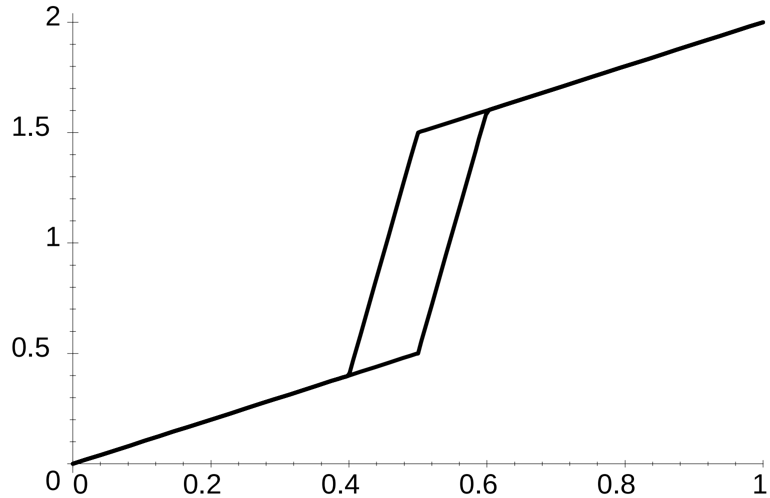

Figure 3: $\varepsilon-$equilibrium strategies

First, we demonstrate that the depicted bidding functions are indeed $\varepsilon-$efficient. Asymptotic efficiency requires that for $t > 1/2$ the probability of winning an object converges to one. For the bidding functions above, the probability that a bidder is bidding above $3/2$ is bounded above by $q(1/2-\varepsilon)+(1-q)(1/2) < 1/2$ and hence any bid above $3/2$ wins the object with probability close to one if r is large. Every bidder with taste parameter $t \in [1/2+\varepsilon, 1]$ bids above $3/2$ and hence almost certainly wins an object. Hence we have asymptotic $\varepsilon-$efficiency.

Second, we demonstrate that the above bidding functions aggregate information. To see this, observe that when the true quality is q, then for given $\hat{q}$, the probability of a bid below $1/2 + \hat{q}$ is

$$\begin{aligned}
\Pr(b \leq 1/2 + \hat{q}|q) &= \pi_1(q)\,(1/2 + \varepsilon\hat{q}) + \pi_2(q)(1/2 - \varepsilon + \varepsilon\hat{q}) \\
&= (1-q)(1/2 + \varepsilon\hat{q}) + q(1/2 - \varepsilon(1-\hat{q})) \\
&= 1/2 + \varepsilon(\hat{q} - q).
\end{aligned}$$

By the law of large numbers this implies that the median bid when quality is q must converge to $1/2 + q$ in probability, yielding information aggregation. Finally, we must demonstrate that the bidding functions constitute an $\varepsilon-$ equilibrium. To see this, observe that since the equilibrium price is approximately $1/2 + q$ for large

n, every bidder in the interval $(1/2-\varepsilon, 1/2+\varepsilon)$ is approximately indifferent between getting the object and not getting the object. Hence the indicated strategies are an $\varepsilon-$best response for those types. Bidders with $t > 1/2 + \varepsilon$ all prefer to receive the object and do so according to the stated bidding strategies. Similarly, bidders with $t < 1/2 - \varepsilon$ do not receive an object, which again is optimal. Hence the bidding functions constitute an $\varepsilon-$equilibrium.

We can strengthen these results and take ε to zero along the sequence while retaining information aggregation (as long as we do so slowly enough that the law of large numbers applies to the number of players with t in $(t^* - \varepsilon_r, t^* + \varepsilon_r)$. Since information aggregation will continue to hold, the strategies are optimal "in the limit". In addition, this sequence of ε_r-equilibria is asymptotically efficient.

The strategy for constructing $\varepsilon-$ equilibria with the desired properties for more general signal structures is the following. First, bidders whose taste is below the value $t^* - \varepsilon$ bid as if $q = 0$ and bidders whose taste is above the value $t^* + \varepsilon$ bid as if $q = 1$. We know from Proposition 4 that this is a feature of all equilibria that aggregate information and are asymptotically efficient. Second, bidders in the interval $t \in [t^* - \varepsilon, t^* + \varepsilon]$ bid *as if* this was a common value auction where all bidders have valuation $t^* + q$. The number of objects in this common value auction is the total number of objects minus the expected number of bidders with valuations above $t^* + \varepsilon$. We know that any symmetric equilibrium in a common value auction aggregates information and that equilibria exist for these auctions (see Pesendorfer and Swinkels (1997)) and hence we know that such strategies can be found. Since information is aggregated and price converges to $q + t^*$ it follows that the bidders in the interval $[t^* - \varepsilon, t^* + \varepsilon]$ are approximately indifferent between receiving the object and not receiving it. Hence, we know that their bids are $\varepsilon-$best responses. Bidders with higher t are playing close to a best response by winning always, and bidders with lower t are playing close to a best response by winning never. Hence we have an ε-equilibrium.[10]

[10] The described equilibrium can be made to be strictly increasing in t by having players use t as their randomizing device for any given s (in Pesendorfer and Swinkels (1997), it is shown that equilibria are non-atomic). It can be also be made continuous by patching things together over the intervals $(t^* - \varepsilon - \delta, t^* - \varepsilon)$ and $(t^* + \varepsilon, t^* + \varepsilon + \delta)$. If δ is chosen small enough relative to ε, the behavior of the players in these intervals does not significantly the expectation of q conditional on any given p being pivotal. And, since information is aggregated, and the players in these two

6. Conclusion

We explore information aggregation and efficiency in a non-cooperative model of price formation where agents have private information and differ in their tastes. While efficiency requires that behavior is a function of taste only, information aggregation can only be accomplished if behavior is also affected by private information.

We show that if there are many agents the market can accomplish both an efficient allocation and aggregate information: as the market grows each participant pays less and less attention to his private information, so that in the limit, allocations are efficient. At the same time the information embodied in *aggregate* behavior becomes more informative, and, in the limit, reflects the true state of the world.

We demonstrate these results in a model that is quite special in many dimensions. However, the ideas driving our results seem quite robust and may be summarized as follows:

Efficiency: a non-negligible misallocation of resources implies that there are many agents who both receive information and act on that information in a non-trivial way. But then, aggregate behavior, as reflected in price, must be very informative about quality. Hence it cannot be optimal for many agents to act on their information.[11]

Information aggregation: In the case of a fixed signal structure the price must make individual signals almost redundant. Otherwise, each of a large number of market participants has an incentive to act on their private information and hence the price will reflect that information.[12]

Our efficiency result holds even if the number of informed bidders is endogenously determined (as for example, with costly signals), and in particular, even if the fraction of informed players drops to zero.

In contrast, we prove the information aggregation result by assuming a fixed signal

intervals have t near t^*, they are playing near optimally.

[11]An critical assumption for this argument is that the information of individual buyers are close substitutes. More precisely, when the information of many buyers is pooled each signal becomes negligible and hence no buyer has information that is essential for assessing the quality of the objects

This is in contrast to an Akerlof type model, where each individual seller has a monopoly over information pertaining to his endowment. The addition of more sellers does not diminish this monopoly power and inefficiency results.

[12]Whether or not information is aggregated in our model is irrelevant for welfare. However, it is straightforward to extend the model in such a way that failure of information aggregation leads to welfare losses. Such extensions are discussed in Pesendorfer and Swinkels (1997) p. 1265.

structure. While this assumption can be relaxed somewhat, there is a real difference between what is needed for efficiency vs. what is needed for information aggregation. This is best seen in the context of costly information acquisition, where efficiency holds but price must remain a noisy signal of quality for there to be an incentive for some players to purchase information. Thus, in the costly signal case, the fraction of informed bidders must not only go to zero, but do so quickly enough to preclude full information aggregation.

One implication is that we need to be careful in rational expectations general equilibrium models as to how we describe the amount of information in price. In particular, there are examples (see Pesendorfer and Swinkels (1997, Section 3.5)) in which a vanishing fraction of informed bidders is enough for information aggregation. On the other hand, in the costly information case, we have a situation where a zero measure set of agents is informed and information is not fully aggregated. The continuum limit cannot distinguish between these two cases. The "correct" price function is perfectly revealing in one case and not in the other. Thus, the appropriate equilibrium price function may depend rather subtly on the details of the convergence to the limit.

7. Proofs

7.1. Proof of Proposition 1

The following observations will be used in the proofs.

1. Since f, w are strictly positive continuous functions on a compact set there is a $\gamma > 0$ such that $f > \gamma, w > \gamma$.

2. Since $\frac{\partial}{\partial q}u(q,t) > 0, \frac{\partial}{\partial t}u(q,t) > 0$ for all $(q,t) \in [0,1] \times [0,\bar{t}]$ and since the partial derivatives are continuous functions it follows that there is an $\eta > 0$ such that the indifference curves defined by

$$u(q,t) = c$$

have slopes bounded by $-\eta$ and $-1/\eta$.

The following notation is used in the subsequent proofs. Let $K(b|s)$ denote the probability that a bid b wins in equilibrium given that the bidder has received signal s. Let $\Pi(b,t,s)$ denote the payoff of a bidder with taste t and signal s if he bids b. Let H_s denote an equilibrium bid distribution, i.e., H_s is a measure on $[0,\bar{t}]\times[0,\infty)$ with marginal W on its first coordinate. For a (Lebesgue) measurable subset $X\subset[0,\infty)$ let $H_s(X)$ be the probability of a bid $b\in X$ by a player with signal s.

For a given auction and equilibrium, and for every bid made with signal s, define the function $t_s(b)$ by $b(s,t_s(b))=b$. If b is larger than any bid made, set $t_s(b)=\bar{t}$, and similarly $t_s(b)=0$, for b smaller than any bid made. Since $b(t,s)$ is a strictly increasing continuous function, $t_s(b)$ is unique. Note that $H_s([0,b])=W(t_s(b))$. Note also that $\mathcal{H}_s\equiv[b(0,s),b(\bar{t},s)]$ is the support of H_s.

By Assumption 1, $t_s(b)$ is increasing and differentiable at all but finitely many points. Hence $t_s(b)$ is absolutely continuous and strictly increasing. Let $\mathcal{H}=\cup_s\mathcal{H}_s$, and let $\mathcal{B}$ be the subset of all $b\in\mathcal{H}$ for which all $t_s(b)$ are differentiable and for which $t_s'(b)\neq 0$ for at least one s. Since $t_s(b)$ is absolutely continuous, $H_s(\mathcal{B})=1, s\in\{1,\ldots,S\}$.

Clearly if for all s, $t_s(.)$ is differentiable at b, then $K(b|s)$ is also differentiable at b, so that $K(b|s)$ is differentiable for all $b\in\mathcal{B}$. If $t_s^{'}(b)>0$ for some s then $K'(b|s)>0$. Since $H_s(\mathcal{B})=1,\forall s$, it follows that $\text{supp}K(.|s)=\mathcal{H}$. Finally, note that for $b\in\mathcal{B}$, $E\left(u(q,t)\,|d=b,s\right)$ is well defined. To see this, note that at points at which all the functions $t_s(.)$ are differentiable, and at least one derivative is strictly positive,

$$\lim_{\varepsilon\to 0}E\left(u(q,t)\,|d\in(b-\varepsilon,b+\varepsilon),s\right)$$

is well defined and unique.

Lemma 1. *If $b\in\mathcal{H}_s\cap\mathcal{B}$ then $E[u(q,t_s(b))|d=b,s]=b$.*

Proof Consider a point $b\in\mathcal{H}_s\cap\mathcal{B}$. Since $b\in\mathcal{B}$ it follows that $E\left(u(q,t)\,|d=b,s\right)$ and $K'(b|s)$ are well defined and hence $\partial\Pi(b,t,s)/\partial b$ is well defined and

$$\frac{\partial\Pi(b,t,s)}{\partial b}=\left[E[u(q,t)|d=b,s]-b\right]K'(b|s).$$

Since $b \in \mathcal{B}, K'(b|s) > 0$. And, since $b \in \mathcal{H}_s$, b is optimal for type $t_s(b)$, and so it follows that $E[u(q, t_s(b))|d = b, s] - b = 0$. ■

The following lemma shows that equilibrium bids are increasing in the signal.

Lemma 2. *For all t, and for all $s' > s$, $b(t, s') > b(t, s)$.*

Proof First consider any bid $b \in \mathcal{B}$ that is made by at least two types of bidder $(t_s(b), s)$ and $(t_{s'}(b), s')$, $s' > s$. In this case

$$E[u(q, t_s(b))|d = b, s] = E[u(q, t_{s'}(b))|d = b, s'] = b.$$

By strict MLRP and since u is strictly increasing in q we have that

$$E[u(q, t)|d = b, s'] > E[u(q, t)|d = b, s] \tag{7.1}$$

for all t, and hence $t_s(b) > t_{s'}(b)$. Since $b(t, s)$ is continuous and strictly increasing in t this further implies that if $b(t, s') > b(t, s)$ for some t then $b(t, s') \geq b(t, s)$ for all t. That is, bidding functions never cross. To see this, assume that for some t', $b(t', s') < b(t', s)$, and, without loss of generality, assume $t < t'$. Then, for bids b in a neighborhood of $b(t, s')$, $t_{s'}(b) < t_s(b)$, while for bids in a neighborhood of $b(t', s')$, $t_{s'}(b) > t_s(b)$ (draw a picture). But, each of these neighborhoods must have non-empty intersection with $\mathcal{B}$, and hence one of these two inequalities is inconsistent with (7.1).

Thus, we may order the signals according to the induced bidding behavior. Further, the only way in which the lemma could fail is if for some $s' > s$, $b(., s')$ lay everywhere below $b(., s)$, and in addition, $b(., s')$ and $b(., s)$ have non-overlapping supports, so that (7.1) does not bind. Let $(s_1, ..., s_S)$ be a permutation of $(1, ..., S)$ such that $b(t, s_j) \geq b(t, s_k), \forall t$ whenever $s_j > s_k$, and assume that $(s_1, ..., s_S) \neq (1, ..., S)$, so that the lemma fails. Let s_k be the largest k such that $s_k < k$. Then, since $s_{k'} \geq s$ for all $k' > k$, a simple induction shows that $s_{k'} = k'$ for all $k' > k$ (begin by noting that $s_S \geq s$ implies $s_S = s$). Thus, all signals in $\{k + 1, \ldots, S)$ are used up by elements of the permutations after k. Hence all bidders with signals $s > k$ bid above $b(0, s_k)$. And, since k does not appear later in the permutation, it must be that k appears before

s_k in the permutation, and so $b(t,k) \leq b(t,s_k)$ for all t. Moreover, since supports of misordered signals are non-overlapping, it must be the case that $b(\bar{t},k) \leq b(0,s_k)$.

Now consider a deviation of type $\bar{t},k$ to the bid $b(\bar{t},s_k)$. Every bid in the interval $(b(\bar{t},k), b(0,s_k))$ is either made by no bidder or by a bidder with type $(t,s), t \leq \bar{t}, s < k$. By Proposition 1 this implies that for all $b \in (b(\bar{t},k), b(0,s_k)) \cap \mathcal{B}$

$$E[u(q,\bar{t})|d=b,k] > E[u(q,t_s(b))|d=b,s'] = b.$$

Similarly, for all $b \in (b(0,s_k), b(\bar{t},s_k)) \cap \mathcal{B}$

$$E[u(q,\bar{t})|d=b,k] > E[u(q,t_{s_k}(b))|d=b,s_k] = b.$$

Hence the deviation is strictly profitable if there is a positive mass of bidders bidding in $(b(\bar{t},k), b(\bar{t},s_k)) \cap \mathcal{B}$. But, in particular, all bidders who receive the signal s_k bid in $[b(\bar{t},k), b(\bar{t},s_k)]$ and since each signal is received by a strictly positive fraction of types (by A1.4), we are done. ■

7.2. Proof of Proposition 2

Let

$$\begin{aligned} Y_r(b|q) &= \sum_s \int_0^{t_s(b)} \pi_r(s|q,t)w(t)dt \\ &= \int_0^1 \sum_{\{s:b_r(t,s)\leq b\}} \pi_r(s|q,t)w(t)dt \end{aligned}$$

denote the probability that a bidder bids below b given q in auction r. Observe that since $b_r(t,s)$ is increasing in s it follows that $\sum_{\{s:b_r(t,s)\leq b\}} \pi_r(s|q,t)w(t)$ is non-increasing in q. Hence $Y_r(b|q)$ is non-increasing in q.

Lemma 3. *The auction is asymptotically efficient if* $Y_r(b|0) - Y_r(b|1) \to 0$ *uniformly for all* b *as* $r \to \infty$.

Proof Observe that $Y_r(b|0,t)-Y_r(b|1,t) \geq 0$, for all t. Hence, $Y_r(b|0)-Y_r(b|1) \to 0$ implies that for all $\varepsilon > 0$ there is an $\bar{r}$ such that for $r > \bar{r}$ there are sets $T_r \subset [0,\bar{t}]$ with $W(T_r) > 1-\varepsilon$ and

$$Y_r(b|0,t) - Y_r(b|1,t) \leq \varepsilon$$

for $t \in T_r$. This in turn implies that

$$\sum_{\{s:b_r(t,s)\leq b\}} (\pi_r(s|0,t) - \pi_r(s|1,t)) \leq \varepsilon \tag{7.2}$$

for all $t \in T_r$ along the subsequence.

Now observe that by Assumption 1 (strict MLRP and $\pi_1(s|q) > \eta$ for all (s,q)) it follows that for all $s' < S$, $\sum_1^{s'} \pi_r(s|q,t)$ is strictly decreasing in q whenever $0 < \sum_1^{s'} \pi_r(s|q',t) < 1$ for some q'. To see this, note that $0 < \sum_1^{s'} \pi_r(s|q',t) < 1$ implies that the relative weight $\pi_{2r}(.|t)$ puts on $s \in \{1,\ldots,s'\}$ and on $\{s'+1,\ldots,S\}$ is bounded away from 0 and ∞. A δ movement between $\{1,\ldots,s'\}$ and $\{s'+1,\ldots,S\}$ relative to $\pi_1(s|q)$ thus translates into a δ' movement between $\{1,\ldots,s'\}$ and $\{s'+1,\ldots,S\}$ relative to $\pi_r(s|q,t) = \pi_1(s|q)\pi_{2r}(s|t)$. Therefore, for all $\delta > 0$, there is $\delta' > 0$ such that if t satisfies

$$\delta < \sum_{\{s:b_r(t,s)\leq b\}} \pi_r(s|q',t) < 1-\delta$$

for some q',then we have that

$$\sum_{\{s:b_r(t,s)\leq b\}} \pi_r(s|0,t) > \sum_{\{s:b_r(t,s)\leq b\}} \pi_r(s|1,t) + \delta'.$$

Therefore, since ε can be chosen arbitrarily small, inequality (7.2) can only hold if either

$$\sum_{\{s:b_r(t,s)\leq b\}} \pi_r(s|0,t) \to 1$$

or

$$\sum_{\{s:b_r(t,s)\leq b\}} \pi_r(s|1,t) \to 0.$$

Hence we may conclude that T_r can be partitioned into two subsets T_r^A and T_r^B where for $t \in T_r^A$ the probability that t wins an object converges to 1 whereas for $t \in T_r^B$ the probability that t wins an object converges to zero.

To prove the lemma it suffices to show that in the limit, T_r^B lies below T_r^A. So, let $t_r^a \in T_r^A$ and $t_r^b \in T_r^B$ be convergent subsequences with limits t^a and t^b respectively. Suppose $t^b > t^a$. Then, for r large enough, $t_r^b > t_r^a$. But then, since bids are increasing in t, $b_r(t_r^b, s) > b_r(t_r^a, s)$ for all $s \in \{1, ..., S\}$. Hence if $t^b > t^a$ then since by definition types with $t \in T_r^B$ win with probability going to 0 while types with $t \in T_r^A$ win with probability going to 1, it must be that the probability that t_r^b receives a signal that is strictly larger than the signal that t_r^a receives converges to one. As this violates the law of iterated expectation, we are done. ■

Denote by $X_r(b)$ the event that $k_r - 1$ of bidders $\{3, \ldots, n_r\}$ bid above b and $n_r - k_r - 1$ bidders bid below b. Because the equilibrium is symmetric, the event $d_r = b$ can be replaced for all relevant purposes by the event $X_r(b) \cap \{b_2 = b\}$. We will use this notation in the following lemma which proves Proposition 2.

Lemma 4. $Y_r(b|0) - Y_r(b|1) \to 0$ *uniformly for all bas* $r \to \infty$.

Proof

Step 1. Assume contrary to the Lemma that along some subsequence

$$Y_r(b_r|0) - Y_r(b_r|1) > \varepsilon \tag{7.3}$$

We first demonstrate that (7.3) implies

$$\Pr[|q - q_r^*| > \delta | X_r(b_r)] \to 0 \tag{7.4}$$

for some q_r^*. Observe that (7.3) implies there is a subset T_r such that for all $t \in T_r$

$$1 \geq Y_r(b_r|0, t_r) - Y_r(b_r|1, t_r) > \varepsilon \tag{7.5}$$

and $\int_{T_r} w(t)dt > \varepsilon$.

Let $S_r^B(t_r) = \{s : b_r(t_r, s) \leq b_r\}$ and let $S_r^A(t_r) = \{1, ..., S\} \backslash S_r^B$. Let $s_r^b = \max S_r^B(t_r)$ and $s_r^a = \min S_r^A(t_r)$. Clearly $s_r^b + 1 = s_r^a$.

Since $\pi_1(s|q) > \eta$ for all (s, q) by Assumption 1, (7.5) implies that there is $\delta(\varepsilon) > 0$ such that for all q

$$\Pr(s \in S_r^B(t_r)|q, t_r) \geq \delta(\varepsilon), \Pr(s \in S_r^A(t_r)|q, t_r) \geq \delta(\varepsilon).$$

Now observe that

$$\begin{aligned}
&\frac{\partial}{\partial q} Y_r(b_r|q, t_r) = \frac{\partial}{\partial q} \Pr(s \in S_r^B(t_r)|q, t_r) \\
= \;& \frac{\partial}{\partial q} \frac{\sum_{s \in S_r^B(t_r)} \pi_r(s|q, t_r)}{\sum_{s \in S_r^B(t_r)} \pi_r(s|q, t_r) + \sum_{s \in S_r^A(b_r)} \pi_r(s|q, t_r)} = \frac{\partial}{\partial q} \frac{\frac{\sum_{s \in S_r^B(t_r)} \pi_r(s|q,t_r)}{\sum_{s \in S_r^A(t_r)} \pi_r(s|q,t_r)}}{\frac{\sum_{s \in S_r^B(t_r)} \pi_r(s|q,t_r)}{\sum_{s \in S_r^A(t_r)} \pi_r(s|q,t_r)} + 1} \\
= \;& \left(\frac{1}{\frac{\sum_{s \in S_r^B(t_r)} \pi_r(s|q,t_r)}{\sum_{s \in S_r^A(t_r)} \pi_r(s|q,t_r)} + 1} \right)^2 \frac{\partial}{\partial q} \left(\frac{\pi_1(s_r^b|q)}{\pi_1(s_r^a|q)} \frac{\sum_{s \in S_r^B(t_r)} \frac{\pi_1(s|q)}{\pi_1(s_r^b|q)} \pi_{2r}(s|t_r)}{\sum_{s \in S_r^A(t_r)} \frac{\pi_1(s|q)}{\pi_1(s_r^a|q)} \pi_{2r}(s|t_r)} \right) \\
\leq \;& \left(\frac{\partial}{\partial q} \frac{\pi_1(s_r^b|q)}{\pi_1(s_r^a|q)} \right) \left(\frac{1}{\frac{\sum_{s \in S_r^B(t_r)} \pi_r(s|q,t_r)}{\sum_{s \in S_r^A(t_r)} \pi_r(s|q,t_r)} + 1} \right)^2 \left(\frac{\sum_{s \in S_r^B(t_r)} \frac{\pi_1(s|q)}{\pi_1(s_r^b|q)} \pi_{2r}(s|t_r)}{\sum_{s \in S_r^A(t_r)} \frac{\pi_1(s|q)}{\pi_1(s_r^a|q)} \pi_{2r}(s|t_r)} \right) \\
\leq \;& \frac{\partial}{\partial q} \frac{\pi_1(s_r^b|q)}{\pi_1(s_r^a|q)} \left(\frac{\delta(\varepsilon)}{2} \right)^2 \eta \delta(\varepsilon) \equiv \gamma(\varepsilon) < 0
\end{aligned}$$

where the first inequality follows from MLRP and the fact that s_r^b is the largest signal in the set $S_r^B(t_r)$ and conversely s_r^a is the smallest signal in the set $S_r^A(t_r)$. Thus, (7.3) implies that

$$\frac{\partial}{\partial q} Y_r(b_r|q) \leq \gamma(\varepsilon) \cdot \varepsilon < 0. \tag{7.6}$$

Let

$$\alpha_r(q) = Y_r(b_r|q)^{\frac{n_r - k_r - 1}{n_r - 2}} [1 - Y_r(b_r|q)]^{\frac{k_r - 1}{n_r - 2}}.$$

Let $q_r^* = \arg\max_q \alpha_r(q)$. Let q', q be such that $q' + \epsilon < q < q_r^*$. From (7.6) we know that $|q' - q| \geq \epsilon$ implies that $|Y_r(b_r|q') - Y_r(b_r|q)| \geq k(\varepsilon)$. Since $\alpha_r(q)$ is a single peaked function of q that is uniformly continuous in the exponent it follows that $|\alpha_r(q') - \alpha_r(q)| \geq k'(\epsilon)$ for some $k'(\varepsilon) > 0$.

Now let $f(q|X_r(b_r))$ denote the density of q conditional on $X_r(b_r)$:

$$f(q|X_r(b_r)) = \frac{f(q)\alpha_r(q)^{n_r-2}}{\int f(w)\alpha_r(w)^{n_r-2}dw}.$$

But this implies that,

$$\frac{f(q'_r|X_r(b_r))}{f(q_r|X_r(b_r))} = \frac{f(q')}{f(q)}\left(\frac{\alpha_r(q')}{\alpha_r(q)}\right)^{n_r-2} \leq (1-k'(\epsilon))^{n_r-2}\,\eta^{-2}.$$

An analogous argument holds for q', q such that $q' - \epsilon > q > q_r^*$.

Thus for all $\delta > 0$

$$\Pr[|q - q_r^*| > \delta | X_r(b_r)] \to 0$$

and the proof of Step 1 is complete.

Step 2. $\Pr[|q - q_r^*| > \delta|X_r(b_r)] \to 0$ implies that $t_{sr}(b_r) - t_{s'r}(b_r) \to 0$ whenever $b_r \in \mathcal{B}_r \cap \mathcal{H}_{sr} \cap \mathcal{H}_{s'r}$.

Since

$$\Pr[|q - q_r^*| > \delta | X_r(b_r)] \to 0$$

it follows that for r large enough

$$\begin{aligned}
u(q_r^* + 2\delta, t) &\geq E[u(q,t)|X_r(b_r), s_2 = S, s_1 = S] \\
&\geq E[u(q,t)|X_r(b_r), b_2 = b_r, s_1 = s] \\
&\geq E[u(q,t)|X_r(b_r), s_2 = 1, s_1 = 1] \\
&\geq u(q_r^* - 2\delta, t)
\end{aligned}$$

where the first and last inequalities follow from the assumption that individual signals are only boundedly informative (A1.3) and from the uniform continuity of u. Recall that the slopes of the indifference curves of u are bounded above by $1/\eta$. Hence for

$t \leq \bar{t} - 4\delta/\eta$ and for all signals $s' > s$

$$E[u(q,t)|d_r = b_r, s_1 = s'] \leq E[u(q, t + 4\delta/\eta)|d_r = b_r, s_1 = s].$$

Since by the hypothesis of Step 2, $b_r \in \mathcal{B}_r \cap \mathcal{H}_{sr} \cap \mathcal{H}_{s'r}$, its follows that

$$b_r = E[u(q, t_{sr}(b))|d_r = b_r, s_1 = s] = E[u(q, t_{s'r}(b))|d_r = b_r, s_1 = s'].$$

Thus $t_{sr}(b_r) - t_{s'r}(b_r) \leq 4\delta/\eta$ and since δ is arbitrary the claim follows.

Step 3. Say that $\mathcal{H}_{sr}$ and $\mathcal{H}_{s'r}$ *overlap* if their intersection contains an open interval. For $T \subset \{1, \ldots S\}$, let

$$Y_r(b|q,T) = \sum_{s \in T} \int_0^{t_s(b)} \pi_r(s|q,t) w(t) dt.$$

Claim: If $\mathcal{H}_{sr}$ and $\mathcal{H}_{s'r}$ overlap for all r along a subsequence then

$$\Pr[|q - q_r^*| > \delta | X_r(b_r)] \to 0$$

implies that

$$Y(b_r|0, \{s, s'\}) - Y(b_r|1, \{s, s'\}) \to 0.$$

Thus, the probability that a bidder bids below b_r and receives a signal in the set $\{s, s'\}$ converges to zero.

From Step 2, the claim is true for any subsequence with $b_r \in \mathcal{B}_r \cap \mathcal{H}_{sr} \cap \mathcal{H}_{s'r}$. Since $\mathcal{B}_r$ is dense in $\mathcal{H}_{sr} \cap \mathcal{H}_{s'r}$, the result for $b_r \in \mathcal{H}_{sr} \cap \mathcal{H}_{s'r}$ follows from the continuity of t_{sr} and Y_r. Assume w.l.o.g. that $s' > s$, so that $\mathcal{H}_{sr} \cap \mathcal{H}_{s'r} = [\min \mathcal{H}_{s'r}, \max \mathcal{H}_{sr}]$, and consider $b_r > \max \mathcal{H}_{sr}$. Then, $t_{sr}(b_r) = \bar{t}$. Furthermore,

$$\begin{aligned} & t_{sr}(b_r(\bar{t}, s)) - t_{s'r}(b_r(\bar{t}, s)) \\ = \; & \bar{t} - t_{s'r}(b_r(\bar{t}, s)) \\ > \; & t_{sr}(b_r) - t_{s'r}(b_r). \end{aligned}$$

Now, we distinguish two cases. First, consider the case where the probability that a bidder receives signal s' *and* bids above $b_r(\bar{t}, s)$ converges to zero. In this case, the claim is trivially true since $Y_r(b_r|q, \{s, s'\})$ converges to one for all q.

Second, consider a subsequence where the probability that a bidder receives signal s' and bids above $b_r(\bar{t}, s)$ stays bounded away from zero. In that case, $Y_r(b_r(\bar{t}, s)|0, \{s, s'\})-Y_r(b_r(\bar{t}, s)|1, \{s, s'\})$ stays bounded away from zero and (by MLRP) this implies that $Y_r(b_r(\bar{t}, s)|0) - Y_r(|1)$ stays bounded away from zero. Now since $b_r(\bar{t}, s) \in \mathcal{H}_{sr} \cap \mathcal{H}_{s'r}$ it follows from the argument above that $t_{sr}(b_r(\bar{t}, s)) - t_{s'r}(b_r(\bar{t}, s)) \to 0$ and hence

$$0 \leq t_{sr}(b_r) - t_{s'r}(b_r) < t_{sr}(b_r(\bar{t}, s)) - t_{s'r}(b_r(\bar{t}, s)) \to 0.$$

and hence $t_{sr}(b_r) - t_{s'r}(b_r) \to 0$ and the claim follows.

An analogous argument handles the case $b_r < \min \mathcal{H}_{s'r}$.

Step 4. For each r, group signals together in such a way that s and s' are in the same group if and only if there is a chain of signals $s = s_1, \ldots, s_k = s'$ with the property that s_i and s_{i+1} have overlapping supports for $i = 1, \ldots, k-1$. This defines a partition of the set of signals which we denote by $\{G_{jr}\}_{j=1}^{K_r}$. As a corollary of Step 3 we may conclude that for all j,

$$Y_r(b_r|0, G_{jr}) - Y_r(b_r|1, G_{jr}) \to 0$$

whenever

$$\Pr[|q - q_r^*| > \delta | X_r(b_r)] \to 0.$$

This follows from repeated application of step 3, noting that the chain separating two members in a group cannot have more than S members. Thus we may summarize Steps 1-3 by concluding that for all j

$$Y_r(b|0, G_{jr}) - Y_r(b|1, G_{jr}) \to 0 \tag{7.7}$$

uniformly in b. This follows since Steps 1-3 have established that for any sequence with the property that

$$Y_r(b_r|0) - Y_r(b_r|1) > \varepsilon$$

(7.7) holds. On the other hand, if

$$Y_r(b_r|0) - Y_r(b_r|1) \to 0$$

(7.7) trivially follows.

Step 5. For each r, let G_r^* be a group for which $\Pr(s \in G)$ is maximal over the set of groups. Then, $\Pr(s \in G_r^*) \to 1$.

Assume not, so that for some $\varepsilon > 0$, in each r along a subsequence, there are at least two groups of signals receiving weight at least ε. Then, for each r in the subsequence there exist consecutive signals s_r and s_r' with the property that s_r and s_r' have non-overlapping support, and such that the probability of a signal of s_r or lower and of a signal of s_r' or greater is at least ε. To see this, order the groups by the signals they contain (clearly if k and k' are in a group, then so are signals between k and k'), and let G_r be the lowest group which receives weight at least ε. Then, s_r can be chosen as the largest element of G_r, and s_r' as $s_r + 1$.

Since s_r and s_r' have non-overlapping support, $b(\bar{t}, s_r) \leq b(0, s_r')$, and so by Proposition 1, no one bids in the interval $(b(\bar{t}, s_r), b(0, s_r'))$. It follows that $X_r(b(\bar{t}, s_r))$ and $X_r(b(0, s_r'))$ are the same event: each occurs when exactly $k_r - 1$ of bidders $3, \ldots, n_r$ received signal s_r' or greater.

But, because there is probability at least ε of signals both below s_r, and above s_r', it follows (by an argument very similar to step 1) that for large r, the information in $X_r(b(\bar{t}, s_r))$ (or equivalently $X_r(b(0, s_r'))$) is enough to tie down q very precisely. But then s_r and s_r' cannot be very informative about q, which, given that $b(\bar{t}, s_r) \leq b(0, s_r')$, contradicts that

$$b(\bar{t}, s_r) = E(u(q, \bar{t})|d_r = b(\bar{t}, s_r), s_r)$$

and

$$b(0, s_{r'}) = E(u(q, 0)|d_r = b(0, s_{r'}), s_{r'})$$

both hold.[13]

Step 6. $Y_r(b|0) - Y_r(b|1) \to 0$ uniformly for all b.

This follows since, by Step 5, almost all signals fall in a single group in the limit, while by Step 4, $Y_r(b|0, G_j) - Y_r(b|1, G_j) \to 0$ uniformly in b for every group of signals. ■

7.3. Proof of Proposition 3

Let $\bar{b}_{sr}$ denote the highest bid made with signal s, and let $\underline{b}_{sr}$ denote the lowest bid with signal s. The following lemma shows that the pivotal bid d_r falls in the interval $[\underline{b}_{Sr}, \bar{b}_{1r}]$ with probability close to one if r is large and k_r/n_r is bounded away from zero and one. Thus the pivotal bid almost always falls in the range of bids where both types with signal 1 and types with signal S bid.

Lemma 5. *Suppose Assumptions 1-3 hold and that* $\pi(s|q)$ *is fixed along the sequence of auctions. Then,* $\Pr\{d_r \in [\underline{b}_{Sr}, \bar{b}_{1r}]\} \to 1$ *as* $r \to \infty$.

Proof: We begin by establishing a property of MLRP. Let $1 \leq s' < S$, let $T = \{1, \ldots s'\}$, and let $T^c = \{s' + 1, \ldots, S\}$. Then,

Claim: $\pi(T|0) - \pi(T|1) \geq \min\left[\pi(1|0) - \pi(1|1), \pi(S|1) - \pi(S|0)\right].$

Proof of claim: This is trivial if $s' = 1$ or $s' = S - 1$. Assume not, and define $\hat{T} = T\backslash 1$, and $\hat{T}^c = T^c\backslash S$. Note that,

$$\begin{aligned}
\frac{\pi(\hat{T}^c|1)}{\pi(\hat{T}|1)} &= \frac{\pi(\hat{T}^c|0) + (\pi(T^c|1) - \pi(T^c|0)) - (\pi(S|1) - \pi(S|0))}{\pi(\hat{T}|0) + (\pi(1|0) - \pi(1|1)) - (\pi(T^c|1) - \pi(T^c|0))} \\
&\geq \frac{\pi(\hat{T}^c|0)}{\pi(\hat{T}|0)}.
\end{aligned}$$

where the equality is accounting, and the inequality is by MLRP.

Cross multiplying and simplifying yields

$$\left(\pi(\hat{T}^c|0) + \pi(\hat{T}|0)\right)(\pi(T^c|1) - \pi(T^c|0))$$

[13] It is possible that one or both of $b(\bar{t}, s_r)$ and $b(0, s_r')$ are not elements of $\mathcal{B}$. However, noting that all relevant entities are continuous, one can work instead with bids near $b(\bar{t}, s_r)$ and $b(0, s_r')$ to derive the same contradiction.

$$\geq \pi(\hat{T}^c|0)\left(\pi(1|0)-\pi(1|1)\right)+\pi(\hat{T}|0)\left(\pi(S|1)-\pi(S|0)\right)$$
$$\geq \left(\pi(\hat{T}^c|0)+\pi(\hat{T}|0)\right)\min\left[\pi(1|0)-\pi(1|1),\pi(S|1)-\pi(S|0)\right]$$

from which

$$\pi(T^c|1)-\pi(T^c|0)\geq \min\left[\pi(1|0)-\pi(1|1),\pi(S|1)-\pi(S|0)\right]$$

and we are done.

This in hand, note that $(1-W(t_{rs}(\bar{b}_{1r}))$ denotes the probability that a bidder bids above $\bar{b}_{1r}$ when he receives signal s. Recall that $\pi(s|q)$ is bounded away from zero for all q, $\pi(S|q)$ is strictly increasing in q, and $\pi(1|q)$ is strictly decreasing in q. It therefore follows from the claim that

$$\begin{aligned}&Y_r(\bar{b}_{1r}|0)-Y_r(\bar{b}_{1r}|1) \\ \geq\ &\min\left[\pi(1|0)-\pi(1|1),\pi(S|1)-\pi(S|0)\right]\left(1-W(t_{rS}(\bar{b}_{1r}))\right).\end{aligned} \tag{7.8}$$

To see this, note that for each t, the set of signals who bid at or below $\bar{b}_{1r}$ is a set of the form $\{1,\ldots,s'\}$, where $s'\geq 1$ always (by definition of $\bar{b}_{1r}$), and $s'<S$ whenever $t>t_{rS}(\bar{b}_{1r})$. Conditional on such a t, $Y_r(\bar{b}_{1r}|0,t)-Y_r(\bar{b}_{1r}|1,t)>\min\left[\pi(1|0)-\pi(1|1),\pi(S|1)-\pi(S|0)\right]$ using the claim. Since $t>t_{rS}(\bar{b}_{1r})$ holds $1-W(t_{rS}(\bar{b}_{1r}))$ of the time, we are done.

Hence, since by Lemma 4

$$Y_r(\bar{b}_{1r}|0)-Y_r(\bar{b}_{1r}|1)\to 0, \tag{7.9}$$

it follows that

$$W(t_{rS}(\bar{b}_{1r}))\to 1.$$

A similar argument shows that

$$W(t_{r1}(\underline{b}_{Sr}))\to 0.$$

For $0 \leq x \leq 1$, let t_x satisfy $W(t_x) = x$. By the strong law of large numbers the fraction of bidders with $t < t_x$ converges to x in probability as $r \to \infty$. Now observe that $\Pr\{d_r < \underline{b}_{Sr}\}$ is equal to the probability that at least an $\frac{n_r - k_r}{n_r}$ fraction of $t's$ drawn falls in the interval $[0, t_{r1}(\underline{b}_{Sr})]$. As long as k_r/n_r stays bounded away from zero and one (Assumption 3) this probability converges to zero (by the law of large numbers). A analogous argument shows that $\Pr\{d_r > \bar{b}_{1r}\} \to 0$ and hence the lemma follows. ■

The proof of Proposition 3 proceeds in two parts. The following lemma shows that if b is the pivotal bid then, for large r, there is essentially no uncertainty about the value of q. Let $\bar{q}_r = E(q|d_r = b)$.

Lemma 6. *Suppose Assumptions 1-3 hold and that $\pi(s|q)$ is fixed along the sequence of auctions. For all $\delta > 0$ there is an $\bar{r}$ such that for all $r > \bar{r}$, $\Pr\{|q - \bar{q}_r| > \delta | d_r = b\} < \delta$ for every $b \in [\underline{b}_{Sr}, \bar{b}_{1r}] \cap \mathcal{B}_r$.*

Proof Let $b \in [\underline{b}_{Sr}, \bar{b}_{1r}] \cap \mathcal{B}_r$. From Proposition 1 we have that

$$E[u(q, t_{r1}(b))|d_r = b, 1) = b$$

and

$$E[u(q, t_{rS}(b))|d_r = b, S) = b.$$

Recall that $\pi(s|q)$ is bounded away from zero for all q and s, $\pi(S|q)$ is strictly increasing in q and $\pi(1|q)$ is strictly decreasing in q. Using the property of MLRP established in the proof of Lemma 5, it therefore follows that for any $b \in [\underline{b}_{Sr}, \bar{b}_{1r}]$

$$\begin{aligned} & Y_r(b|0) - Y_r(b|1) \\ \geq \ & \min\left[\pi(1|0) - \pi(1|1), \pi(S|1) - \pi(S|0)\right]\left(W(t_{r1}(b)) - W(t_{rS}(b))\right) \end{aligned} \qquad (7.10)$$

If the Lemma is false then there is a sequence of bids $b_r \in [\underline{b}_{Sr}, \bar{b}_{1r}]$ such that $\Pr\{|q - \bar{q}_r| > \delta | d_r = b\} \geq \delta$ for all r and for some $\delta > 0$. We will show a contradiction.

Since $Y_r(b_r|0) - Y_r(b_r|1) \to 0$ (by Lemma 4) it follows that $t_{r1}(b_r) - t_{rS}(b_r) \to 0$. Since u is uniformly continuous this in turn implies that

$$E[u(q,t)|d_r = b_r, S) - E[u(q,t)|d_r = b_r, 1) \to 0. \tag{7.11}$$

Let F_r denote the sequence of probability distributions of q conditional on $d_r = b$. Note that every sequence of probability distributions has a convergent subsequence. Consider any convergent subsequence of F_r with limit $\hat{F}$. If $\hat{F}$ is non-degenerate then it follows that

$$\lim \left(E[u(q,t)|d_r = b_r, S) - E[u(q,t)|d_r = b_r, 1)\right) > 0$$

contradicting (7.11). Hence it must be that $\hat{F}$ is a point mass at q and hence

$$\Pr(|q - E(q|d_r = b_r)| > \delta|\, d_r = b_r) \to 0.$$

establishing the desired contradiction.

■

Proof of Proposition 3 By Lemma 6 it follows that whenever $b \in [\underline{b}_{Sr}, \bar{b}_{1r}] \cap \mathcal{B}_r$, $F(q|d_r = b)$ is concentrated around its expectation for large r. Now consider the probability distribution $F(q|p_r = b)$ of q conditional on the *price* being b in auction r. Note that the price is b if k_r bidders bid above b, one bidder bids b, and $n_r - k_r - 1$ bidders bid below b. Thus $F(q|p_r = b) = F(q|d_r = b, b_1 \geq b)$. Observe that for all $\epsilon > 0$ there is an r' such that for $r > r'$

$$|F(q|d_r = b) - F(q|p_r = b)| \leq \max_{s \in S} |F(q|d_r = b) - F(q|d_r = b, s)| \leq \varepsilon$$

The first inequality follows since b_1 is a garbling of player 1's signal. The second inequality holds because $F(q|d_r = b)$ is arbitrarily concentrated around its mean for large r, and hence adding one additional (noisy) signal only changes the distribution by a very small amount. Since by Lemma 6 $F(q|d_r = b)$ is concentrated around the true value q^* it follows that $F(q|p_r = b)$ is also concentrated around the true value q^*.

Consider a bid $b \in [\underline{b}_{Sr}, \bar{b}_{1r}] \cap \mathcal{B}_r$. If a bidder with type (t, s) made the bid b and if the equilibrium price is equal to b then by the preceding argument

$$p_r \in [t + q^* - (\delta + \epsilon), t + q^* + (\delta + \epsilon)]$$

with probability $1-\epsilon-\delta$, where q^* denotes the true q. Further note that by Proposition 2 it must be the case that for large r the bidder who makes the $k_r + 1^{st}$ highest bid has a t very close to t_r^* with high probability. Thus for r sufficiently large we have that

$$p_r \in [t_r^* + q^* - (2\delta + \epsilon), t_r^* + q^* + (2\delta + \epsilon)]$$

with probability larger than $1 - \epsilon - 2\delta$.

To prove the Proposition it is now sufficient to show that

$$\Pr\{p_r \notin [\underline{b}_{Sr}, \bar{b}_{1r}] \cap \mathcal{B}_r\} \to 0$$

as $r \to \infty$. But $\{p_r \in [\underline{b}_{Sr}, \bar{b}_{1r}] \cap \mathcal{B}_r\}$ denotes the event that the $k_r + 1$-st highest bid of n_r bids is in $[\underline{b}_{Sr}, \bar{b}_{1r}] \cap \mathcal{B}_r$.

As was shown in proving Lemma 5, $W[t_{1r}(\underline{b}_{Sr})] \to 0$ and $W[t_{Sr}\left(\bar{b}_{1r}\right)] \to 1$. Since $\frac{k_r}{n_r}$ is bounded from 0 and 1, so, for large r, is $\frac{k_r+1}{n_r}$. Thus the strong law of large numbers implies that

$$\Pr\{p_r \in [\underline{b}_{Sr}, \bar{b}_{1r}]\} \to 1.$$

Finally, since $H_s(\mathcal{B}_r) = 1$ for all s, it follows that $\Pr\{p_r \notin \mathcal{B}_r\} = 0$ for all r which proves the final claim. ■

7.4. Proof of Proposition 4

We will demonstrate that $b_r(t, S) \to u(0, t)$ uniformly for all $t \leq t^* - \varepsilon$. Since $u(0, t) \leq b_r(t, s) \leq b_r(t, S)$ this proves the result for $t \leq t^* - \varepsilon$.

Consider $t \leq t^* - \varepsilon$, and $b_r(t,S) \in [\underline{b}_{Sr}, \bar{b}_{1r}] \cap \mathcal{B}_r$. By Lemma 6 we know that

$$\Pr\{|q - E(q|d_r = b_r(t,S))| > \delta|\, d_r = b_r(t,S)\} \to 0$$

uniformly for all $b_r(t,S) \in [\underline{b}_{Sr}, \bar{b}_{1r}]$, which in turn implies that

$$\Pr\{|q - E(q|X(b_r(t,S))| > \delta|\, X(b_r(t,S))\} \to 0$$

uniformly for all $b_r(t,S) \in [\underline{b}_{Sr}, \bar{b}_{1r}]$ since the information about q contained in the event $\{d_r = b_{Sr}\}$ differs from the information about q contained in the event $X(b_r(t,S))$ by at most one signal. Thus, uniformly for all $b_r(t,S) \in [\underline{b}_{Sr}, \bar{b}_{1r}]$,

$$f(q|X(b_r(t,S))) \equiv \frac{f(q)\alpha_r(q, b_r(t,S))^{n_r}}{\int_0^1 f(w)\alpha_r(w, b_r(t,S))^{n_r} dw} \tag{7.12}$$

converges to a density that has all its mass concentrated at some $\hat{q}_r$. Recall that f is a single peaked function of q. Since $f(q) > \gamma > 0$ and is bounded, $\hat{q}_r$ must satisfy

$$|\hat{q}_r - \arg\max_q \alpha_r(b_r(t,S), q)| \to 0$$

uniformly for all $b_r(t,S) \in [\underline{b}_{Sr}, \bar{b}_{1r}]$.

By Lemma 4,

$$Y_r(b_r(t,S)|0) - Y_r(b_r(t,S)|1) \to 0 \tag{7.13}$$

and by Proposition 2, there is a $\delta > 0$ such that

$$Y_r(b_r(t,S)|0) \leq W(t^*) - \delta$$

for all q, for $t \leq t^* - \varepsilon$ for sufficiently large r. (To see this last inequality assume that $\lim Y_r(b_r(t^* - \varepsilon, S)|0) \geq W(t^*)$ for some convergent subsequence. Then, all types (t,S) with $t \in [t^* - \varepsilon/2, t^*]$ receive the object with probability close to one when q is near 0 and r is sufficiently large. This violates asymptotic efficiency).

But then for large r, $\frac{n_r-k_r-1}{n_r-2} > Y_r(b_r(t,S)|0)$ and $Y_r(b_r(t,S)|q)$ is strictly decreasing in q. Hence, for $q \in (0,1]$,

$$\begin{aligned} \alpha_r(b_r(t,S),q) &= \left(Y_r(b_r(t,S)|q)^{\frac{n_r-k_r-1}{n_r-2}}[1-Y_r(b_r(t,S)|q)]^{\frac{k_r-1}{n_r-2}}\right) \\ &< \left(Y_r(b_r(t,S)|0)^{\frac{n_r-k_r-1}{n_r-2}}[1-Y_r(b_r(t,S)|0)]^{\frac{k_r-1}{n_r-2}}\right) \end{aligned}$$

and consequently, for r sufficiently large,

$$0 = \arg\max_q \alpha_r(b_r(t,S),q)$$

which implies that

$$\hat{q}_r \to 0.$$

Thus, $b_r(t,S) = E(u(q,t)|X_r(b), b_2 = b, S) \leq E(u(q,t)|X_r(b), S, S) \to u(0,t)$ as $r \to \infty$ for all $b_r(t,S) \in \mathcal{B}_r$ with $t \leq t^* - \varepsilon$. Note that $\mathcal{B}_r$ is dense on the set of bids of types (t,S) with $t \leq t^* - \varepsilon$. Thus we have established that for a dense subset of $[0, t^* - \varepsilon]$, $\lim b_r(t,S) = t$. Continuity of $b_r(t,S)$ now gives the result.

The argument for $t \geq t^* + \varepsilon$ is exactly analogous. ■

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