Harmonic Analysis of Boolean Functions, and applications in CS

Lecture 5

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We continue the proof of the FKN theorem which states:

Theorem 1 *Let $f : \{\pm 1\}^n \to \mathbb{R}$ s.t. $\left\| f^{>1} \right\|_2^2 \leq \varepsilon$. Then f is $(16\varepsilon(1 + o(1)), 1)$-junta.*

Last lecture we showed that it's enough to prove the following lemma:

Lemma 2 *Let $f : \{\pm 1\}^n \to \mathbb{R}$ be a linear function, that is $f = a_0 + \sum_{i=1}^n a_i \chi_i$ s.t. $\sum_{i=0}^n a_i^2 \leq 1$. Assume that $|a_1| \geq |a_2| \geq \cdots \geq |a_n|$. Let $\varepsilon = \|f - \mathrm{sign(f)}\|_2^2$. Then $\|f - (a_0 + a_1\chi_1)\|_2^2 \leq \varepsilon(1 + o_\varepsilon(1)$.*

It is easy to see that $|a_2| \leq \frac{1}{\sqrt{2}}$, since $a_1^2 + a_2^2 \leq 1$ and $|a_1| \geq |a_2|$. In fact we can find a better bound for $|a_2|$ and hence all $|a_i|$ for $i \geq 2$. We do it in the following claim:

Claim 3 *$|a_2| < 10\sqrt{\varepsilon}$.*

Proof Since $|a_2| \leq \frac{1}{\sqrt{2}}$, for each fixed $x_1, x_3, x_4, \ldots, x_n$, there is a setting of x_2 such that $|f(x_1, \ldots, x_n) - \mathrm{sign(f(x_1, \ldots, x_n))}| \geq \frac{1}{3}|\mathrm{a_2}|$.

Indeed, let $X = a_0 + \sum_{i=2}^n a_i x_i$ and assume wlog that $X \geq 0$. If $X \leq 1$, then $|X - |a_2| - \mathrm{sign(X - |a_2|)}| \geq \frac{1}{3}|a_2|$ and otherwise $|X + |a_2| - \mathrm{sign(X + |a_2|)}| \geq \frac{1}{3}|a_2|$.

That is $\Pr\left[|f - \mathrm{sign(f)}| \geq \frac{1}{3|\mathrm{a_2}|}\right] \geq \frac{1}{2}$. Therefore, if $|a_2| > 10\sqrt{\varepsilon}$, then using Markov's inequality we get $\mathbb{E}\left[(f - \mathrm{sign(f)})^2\right] \geq \frac{1}{9}a_2^2 \Pr\left[(f - \mathrm{sign(f)})^2 \geq (\frac{1}{3}a_2)^2\right] \geq \frac{1}{9} \cdot 100\varepsilon \cdot \frac{1}{2} > 5\varepsilon$, contradicting the assumption that $\varepsilon = \|f - \mathrm{sign(f)}\|_2^2$. ■

Now we know that $|a_i| < 10\sqrt{\varepsilon}$ for all $i \geq 2$. But we want to show $\sum_{i\geq 2} a_i^2 \leq \varepsilon(1 + o_\varepsilon(1)$, which is much stronger. Let's try to use argument similar to that in claim 3 and try to bound $\|f - \mathrm{sign(f)}\|_2^2$ when some x_i's are fixed. We know thus far that $a_n^2 \leq a_2^2 < 100\varepsilon$. Let m be the smallest index in $[n]$ s.t. $\sum_{i\geq m} a_i^2 \leq 102\varepsilon$. Write $f = a_0 + a_1\chi_1 + \cdots + a_{m-1}\chi_{m-1} + \sum_{i=m}^n a_i\chi_i$. By the choice of m, $\|\sum_{i=m}^n a_i\chi_i\|_2^2 \leq 102\varepsilon$. Now are going to fix $x_1, \ldots, x_{m-1}$, denote $c = a_0 + \sum_{i=1}^{m-1} a_i x_i$ and say something useful about f in the lemma below:

Lemma 4 *Let $g = c + \sum_{i=m}^n a_i\chi_i$ and $\alpha = \sum_{i=m}^n a_i^2 \leq 102\varepsilon$. Then $\|g - \mathrm{sign(g)}\|_2^2 \geq \alpha(1 + o_\varepsilon(1))$.*

Once we prove this lemma, the main lemma will follow:

$$\begin{aligned}
\varepsilon &= \|f - \text{sign(f)}\|_2^2 \\
&= \mathbb{E}_X\left[(f - \text{sign(f)})^2\right] \\
\text{(The total expectation)} &= E_{x_1,\dots,x_{m-1}}\left[\mathbb{E}_{x_m,\dots,x_n}\left[(f - \text{sign(f)})^2\right]\right] \\
\text{(By lemma 4)} &\geq \mathbb{E}_{x_1,\dots,x_{m-1}}\left[\mathbb{E}_{x_m,\dots,x_n}[\underbrace{c + \sum_{i=m}^{n} a_i\chi_i}_{g} - \text{sign}(\underbrace{c + \sum_{i=m}^{n} a_i\chi_i}_{\text{g}})]\right] \\
&\geq \mathbb{E}_{x_1,\dots,x_{m-1}}\left[\alpha(1 + o_\varepsilon(1))\right] \\
&= \alpha(1 + o_\varepsilon(1))
\end{aligned}$$

We need to show $\sum_{i=2}^{n} a_i^2 \leq \varepsilon(1 + o_\varepsilon(1))$ and what we have is $\sum_{i=m}^{n} a_i^2 \leq \varepsilon(1 + o_\varepsilon(1))$. But if $m > 2$ then using claim 3, $\sum_{i=m-1}^{n} a_i^2 \leq 100\varepsilon + \varepsilon(1 + o_\varepsilon(1)) < 102\varepsilon$, contradicting our choice of m and therefore the lemma 2 follows.

Useful facts In order to prove lemma 4, we will use the following facts:

1. **Chernoff bound**: $\Pr_{x\in\{\pm1\}^n}\left[|\sum \gamma_i x_i| \geq t\right] \leq \exp\left(-\frac{t^2}{2\sum \gamma_i^2}\right)$
2. If X is a positive r.v., then $\mathbb{E}[X] = \int_{t=0}^{\infty} \Pr[X > t]dt$.
3. For any r.v. X with finite second moment $\mathbb{E}[(X-1)^2] \geq \text{Var}[X]$.

Proof of Lemma 4 Note that due to symmetry, we may assume that $c \geq 0$. Moreover, using Chernoff bound if $|c - 1| \geq 1/2$ then $\Pr\left[|\sum_{i=m}^{n} a_i x_i| \geq 1/4\right] < \exp\left(-\frac{1}{32\alpha}\right)$ and using Markov's inequality we get the desired result.

Now we assume that $1/2 < c < 3/2$. Then

$$\begin{aligned}
\|g - \text{sign(g)}\|_2^2 &= \||g| - 1\|_2^2 \\
&\geq \||g| - \mathbb{E}[|g|]\|_2^2 \\
&= \mathbb{V}[|g|] \\
&= \mathbb{E}[g^2] - c^2 + c^2 - \mathbb{E}^2[|g|] \\
&= \sum_{i=m}^{n} a_i^2 + (c + \mathbb{E}[|g|])(c - \mathbb{E}[|g|])
\end{aligned}$$

Now we want to bound $c - \mathbb{E}[|g|] = \mathbb{E}[g] - \mathbb{E}[|g|] = \mathbb{E}[g - |g|] = -2\mathbb{E}[g_-]$:

$$\begin{aligned}
\mathbb{E}[g_-] &= \int_{t=0}^{\infty} \Pr\left[g_- > t\right] dt \\
&\leq \int_{t=0}^{\infty} \Pr\left[\sum_{i=m}^{n} a_i x_i + c < -t\right] dt \\
&= \int_{t=0}^{\infty} \Pr\left[\sum_{i=m}^{n} a_i x_i < -c - t\right] dt \\
&\leq \int_{t=0}^{\infty} \Pr\left[|\sum_{i=m}^{n} a_i x_i| > c + t\right] dt \\
\text{(Chernoff bound+change of variables)} &\leq \int_{t=c}^{\infty} \exp\left(-\frac{t^2}{2\alpha}\right) dt
\end{aligned}$$

In order to bound the integral, we multiply it by $\frac{\alpha}{c}\frac{t}{\alpha}$, and use the fact that $c > 1/2$:

$$\begin{aligned}
\mathbb{E}[g_-] &\leq \frac{\alpha}{c} \int_{t=c}^{\infty} \frac{t}{\alpha} \exp\left(-\frac{t^2}{2\alpha}\right) dt \\
&= -\frac{\alpha}{c} \exp\left(-\frac{t^2}{2\alpha}\right)\Big|_c^{\infty} \\
&= \frac{\alpha}{c} \exp\left(-\frac{c^2}{2\alpha}\right) \\
&= 2\alpha \exp\left(-\frac{1}{16\alpha}\right)
\end{aligned}$$

We get that $\mathbb{E}[g] - \mathbb{E}[|g|]$ is exponentially small in α and therefore $c + \mathbb{E}[|g|] = 2c + \alpha o_\alpha(1) < 4$. And therefore $\|g - \text{sign(g)}\|_2^2 \geq \alpha + 4\alpha o_\alpha(1) = \alpha(1 + o_\alpha(1))$ as required. ■