Harmonic Analysis of Boolean Functions, and applications in CS

Lecture 13

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In the privious lecture we saw the Bonami-Beckner transform and some corollaries. Today we will see some more related theorems, like Freidgut's theorem, KKL and BKKKL.

But rst - a reminder of a corollary we want to use,

Corollary 1 *If* $1 \leq p \leq 2 \leq q < \infty$ *then*

$$\left\| f^{\leq k} \right\|_q \leq (q-1)^{k/2} \left\| f \right\|_2$$

$$\left\| f^{\leq k} \right\|_2 \leq (p-1)^{-k/2} \left\| f \right\|_p$$

Now we can state the following corollary:

Corollary 2 *Let* $g : \{\pm 1\}^n \to \{\pm 1, 0\}$. *Then* $\left\| g^{\leq k} \right\|_2^2 \leq 2^k (\|g\|_2^2)^{4/3}$.

Proof Corollary 1 with $p = \frac{3}{2}$ implies $\left\| g^{\leq k} \right\|_2^2 \leq 2^k \|g\|_{3/2}^2$. Since g gets only the values $\{\pm 1, 0\}$, $\|g\|_{3/2}^{3/2} = \|g\|_2^2$, so we have $2^k (\|g\|_{3/2}^2) = 2^k \left(\|g\|_2^2 \right)^{4/3}$. ■

This corollary helps us in proving an important theorem that says that low in uence functions are juntas,

Theorem 3 (Freidgut '95) *Let* $f : \{\pm 1\}^n \to \{\pm 1\}$ *s.t.* $I(f) \leq k$. *Then* f *is a* $\left(4\varepsilon, \frac{8k^2 2^{\frac{6k}{\varepsilon}}}{\varepsilon^3}\right)$*-junta.*

Proof De ne $t = \frac{\varepsilon^3}{8k^3} 2^{-\frac{6k}{\varepsilon}}$, $J = \{i \in [n] \mid I_i(f) \geq t\}$. Then obviously $|J| \leq \frac{k}{t} = \frac{8k^2 2^{\frac{6k}{\varepsilon}}}{\varepsilon^3}$.

De ne $g = \sum_{S \subseteq J} \hat{f}(S) \chi_S$. It's enough to prove $\|f - g\|_2^2 \leq \varepsilon$, because it implies: $\|f - \text{sign(g)}\|_2^2 \leq 4\varepsilon$, which gives the theorem.

We prove $\|f - g\|_2^2 \leq \varepsilon$ in two parts, rst for low frequencies and then for high frequencies:

$$\begin{aligned}
\sum_{\substack{S\setminus J\neq\emptyset \\ |S|\leq\frac{2k}{\varepsilon}}} \hat{f}(S)^2 &\leq \sum_{\substack{S\setminus J\neq\emptyset \\ |S|\leq\frac{2k}{\varepsilon}}} |S\setminus J|\hat{f}(S)^2 \\
&= \sum_{i\in[n]\setminus J} \left\| f_i^{\leq\frac{2k}{\varepsilon}} \right\|_2^2 \\
&\leq 2^{\frac{2k}{\varepsilon}} \sum_{i\in[n]\setminus J} \left(\|f_i\|_2^2\right)^{\frac{4}{3}} \quad \text{(By corollary 2)} \\
&\leq 2^{\frac{2k}{\varepsilon}} \cdot \max_{i\in[n]\setminus J} \{I_i(f)^{\frac{1}{3}}\} \cdot \sum_i I_i(f) \\
&\leq 2^{\frac{2k}{\varepsilon}} \cdot \max_{i\in[n]\setminus J} \{I_i(f)^{\frac{1}{3}}\} \cdot I(f) \\
&\leq 2^{\frac{2k}{\varepsilon}} \cdot t^{\frac{1}{3}} \cdot k \quad \text{(recall that } J = \{i\in[n] \mid I_i(f)\geq t\}\text{)} \\
&\leq \frac{\varepsilon}{2} \quad \text{(follows from } t = \frac{\varepsilon^3}{8k^3}2^{-\frac{6k}{\varepsilon}}\text{)}
\end{aligned}$$

We now deal with the high frequencies:

$$\begin{aligned}
\sum_{S\setminus J\neq\emptyset |S|>\frac{2k}{\varepsilon}} \hat{f}(S)^2 &\leq \sum_{|S|>\frac{2k}{\varepsilon}} \hat{f}(S)^2 \\
&\leq \frac{\varepsilon}{2k}\sum_S |S|\hat{f}(S)^2 \\
&\leq \frac{\varepsilon}{2k}\cdot k \\
&\leq \frac{\varepsilon}{2}
\end{aligned}$$

From the two inequalities above we have $\|f-g\|_2^2 = \sum_{S\setminus J\neq\emptyset}\hat{f}(S)^2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, as we wanted. ■

The next result is that $\exists\delta_0 > 0$ such that the following holds,

Theorem 4 (KKL) *Let $f : \{\pm 1\}^n \to \{\pm 1\}$, $0 < \delta < \delta_0$. If*

$$I(f) \leq \frac{1}{20}\left(\left(1-\hat{f}(\emptyset)^2\right)\log\frac{1}{\delta}\right) \tag{1}$$

then $\exists i$ s.t. $I_i(f) > \delta$.

If f is constant then the theorem is obvious, so we assume that f is balanced.
Proof The function f is balanced, so $\hat{f}(\emptyset)^2 \neq 1$. De ne $k = \frac{2I(f)}{1-\hat{f}(\emptyset)^2}$. Then:

$$\sum_{|S|>k}\hat{f}(S)^2 \leq \frac{1}{k}\cdot\sum_{|S|>k}|S|\hat{f}(S)^2 \leq \frac{1}{k}\cdot\sum_S |S|\hat{f}(S)^2 = \frac{I(f)}{k} = \frac{1-\hat{f}(\emptyset)^2}{2} \tag{2}$$

Now:

$$\begin{aligned}
\frac{1-\hat{f}(\emptyset)^2}{2} &= 1-\hat{f}(\emptyset)^2 - \frac{1-\hat{f}(\emptyset)^2}{2} \\
&\leq \sum_{|S|>0} \hat{f}(S)^2 - \sum_{|S|>k} \hat{f}(S)^2 \\
&= \sum_{0<|S|\leq k} \hat{f}(S)^2 \\
&\leq \sum_{|S|\leq k} |S|\hat{f}(S)^2 \\
&= \sum_{i\in[n]} \left\| f_i^{\leq k} \right\|_2^2 \\
&\leq 2^k \sum_{i\in[n]} \left(\left\| f_i^{\leq k} \right\| \right)^{\frac{4}{3}} \quad \text{(By corollary 2)} \\
&\leq 2^k \cdot \max_i \{ I_i(f)^{\frac{1}{3}} \} \cdot I(f) \quad \text{(similar to the privious proof)} \\
&\leq \left(\frac{1}{\delta}\right)^{10} \cdot \max_i \{ I_i(f)^{\frac{1}{3}} \} \cdot I(f) \quad \text{(Because } 2^k \leq 2^{\frac{\log \frac{1}{\delta}}{10}} \text{)} \\
&\leq \left(\frac{1}{\delta}\right)^{10} \cdot \max_i \{ I_i(f)^{\frac{1}{3}} \} \cdot \frac{1-\hat{f}(\emptyset)^2}{20} \cdot \log\frac{1}{\delta} \quad \text{(By 1)}
\end{aligned}$$

Therefore:

$$max_i\{I_i(f)\} \geq 8000 \cdot \delta^{\frac{3}{10}} \cdot \left(\frac{1}{\log\frac{1}{\delta}}\right)^3 > \delta$$

■

If $\delta = \frac{\log n}{n}$, then the theorem implies the following two corollaries,

Corollary 5 *If f is balanced and $I(f) < C \log n$, then $\exists i$ s.t. $I_i(f) > \frac{\log n}{n}$.*

Corollary 6 *If f is boolean, balanced and transitive, then $I(f) \geq C \log n$.*

We want to state another interesting result, but to do that we need the following de -nition,

Definition 1 *Let $f : \{\pm 1\}^n \to \{\pm 1\}$.*

$$I_i'(f) = \Pr_{x\setminus i}(f(x) \text{ non constant on } x_i)$$

$$I'(f) = \sum_i I_i'(f)$$

Theorem 7 (BKKKL) *Let $f : \{\pm 1\}^n \to \{\pm 1\}$, $t = E(f)$. Then $\exists i$ s.t. $I_i'(f) \geq C(1-t^2)\frac{\log n}{n}$.*

We will talk more about this theorem in our next class.