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So far, we used elementary techniques. Today we will do something less elementary, for the first time.

Reminders and Preliminaries

Definition 1 *The* L_p-norm *of a function* $f : \{\pm 1\}^n \to \mathbb{R}$ *is*

$$\|f\|_p = (\mathbb{E}[|f(x)|^p])^{1/p}$$

for $1 \le p < \infty$ *and*

$$\|f\|_\infty = \max_{x \in \{\pm 1\}^n} |f(x)|$$

We already proved *monotonicity*: if $p < q$ then $\|f\|_p \le \|f\|_q$.
We also have *continuity in* p: $\lim_{p \to q} \|f\|_p = \|f\|_q$. This includes the case $q = \infty$.

Definition 2 $R : \mathbb{R}^{\{\pm 1\}^n} \to \mathbb{R}^{\{\pm 1\}^n}$ *is a* linear transformation *if for any* $f, g \in \{\pm 1\}^n$ *and* $\lambda \in \mathbb{R}$ *we have*

$$R(f + g) = R(f) + R(g)$$

and

$$R(\lambda f) = \lambda R(f)$$

Definition 3 R *is* p-contractive *if for all* $f \in \mathbb{R}^{\{\pm 1\}^n}$ *we have*

$$\|R(f)\|_p \le \|f\|_p$$

Definition 4 R *is* $p \to q$-hypercontractive *if for all* $f \in \mathbb{R}^{\{\pm 1\}^n}$ *we have*

$$\|R(f)\|_q \le \|f\|_p$$

$p \to p$-hypercontractivity is just p-contractivity. If R is $p \to q$-hypercontractive then for any $p \le p' \le q' \le q$ we have that R is $p' \to q'$-hypercontractive, since, by monotonicity $\|R(f)\|_{q'} \le \|R(f)\|_q \le \|f\|_p \le \|f\|_{p'}$.

Definition 5 *Given an indexed set of real numbers* $A = \{a_S\}_{S \subset [n]}$ *define the transform*

$$T_A(f) = \sum_S a_S \hat{f}(S) \chi_S$$

If for all S we have $|a_S| \leq 1$ then T_A is 2-contractive, but not necessarily p-contractive for $p \neq 2$, as we have seen in the exercise. An interesting question is for which A and which p, q is T_A (hyper)contractive.

Example 1 *If we take $a_S = 1$ if $|S| \leq k$ and 0 otherwise, we get that $T_A(f) = f^{\leq k}$.*

We already used this transformation.

Example 2 *The* Rademacher projection *is the transform*

$$Rad(f) = f^{=1} = \sum_i \hat{f}(i)\chi_i$$

Theorem 6 *For all $2 \leq p$ we have*

$$\|Rad(f)\|_p \leq \sqrt{p-1}\|f\|_2$$

If we replace $\sqrt{p-1}$ by $\sqrt{p}$, and require p to be even, then what we get follows from part 3 of exercise 3.

The Bonami-Beckner Transform

Definition 7 *For $0 \leq \delta \leq 1$ the* Bonami-Beckner Transform *is defined by*

$$T_\delta(f) = \sum_S \delta^{|S|}\hat{f}(S)\chi_S \ .$$

For which δ is T_δ (hyper)contractive?

Claim 8 *T_δ is p-contractive for all $0 \leq \delta \leq 1$ and all $p \geq 1$.*

Proof Consider the transform

$$T'_\delta(f) = \mathbb{E}_{z \sim \mu^{(n)}_{(1-\delta)/2}}[f(zx)]$$

A straightforward computation reveals that

$$T'_\delta(f)(x) = \sum_S \hat{f}(S)\chi_S(x)\mathbb{E}_{z \sim \mu^{(n)}_{(1-\delta)/2}}[\chi_S(z)] = \sum_S \hat{f}(S)\chi_S(x)\delta^{|S|} = T_\delta(f)(x)$$

so these two transforms are one. T'_δ can also be written as

$$T'_\delta(f) = \mathbb{E}_{z \sim \mu^{(n)}_{(1-\delta)/2}}[\sigma_z(f)] \ ,$$

where σ_z is the shift by z on the hypercube.

Since $\|\sigma_z(f)\|_p = \|f\|_p$ for any z and $\|\cdot\|_p$ is convex we have

$$\|T'_\delta(f)\|_p = \|\mathbb{E}_{z \sim \mu^{(n)}_{(1-\delta)/2}}[\sigma_z(f)]\|_p \leq \mathbb{E}_{z \sim \mu^{(n)}_{(1-\delta)/2}}\|\sigma_z(f)\|_p = \|f\|_p$$

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Theorem 9 (Bonami 72', Beckner 73') *For $1 \le p \le q$ and $\delta \le \sqrt{\frac{p-1}{q-1}}$ we have*

$$\|T_\delta(f)\|_q \le \|f\|_p$$

We will not prove this theorem, but it can be done by induction on n. However, even the base case ($n = 1$) is far from trivial. Instead, we will see what can be done with it.

Corollary 10 *If $1 \le 2 \le q < \infty$ then*

$$\|f^{\le k}\|_q \le (q-1)^{k/2}\|f\|_2$$
$$\|f^{\le k}\|_2 \le (p-1)^{-k/2}\|f\|_p$$

Proof We prove the second inequality, the first is similar.

Take $\delta = \sqrt{p-1}$. Then,

$$\|f\|_p \ge \|T_\delta(f)\|_2 = \sqrt{\sum_S \delta^{2|S|}\hat{f}^2(S)} \ge \sqrt{\sum_{|S|\le k} \delta^{2|S|}\hat{f}^2(S)} \ge \sqrt{\sum_{|S|\le k} \delta^{2k}\hat{f}^2(S)} = \delta^k\|f^{\le k}\|_2$$

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Next, we use the corollary to prove some cool stuff about the influence of low degree functions.

Corollary 11 *Let $f : \{\pm1\}^n \to \{\pm1\}$ be a Boolean function of degree at most k. Then for each i either*

$$I_i(f) = 0 \qquad \text{or} \qquad I_i(f) \ge 8^{-k}$$

Proof Define

$$f_i(x) = \frac{f(x) - f(\sigma_i x)}{2} \ .$$

Since f is Boolean, we have $|f_i(x)| = 1$ if $f(x) \ne f(\sigma_i x)$ and 0 otherwise. Therefore, for every $1 \le p$

$$\|f_i\|_p^p = I_i(f) \ .$$

By corollary 10, taking $p = 3/2$, we have

$$\|f_i\|_2 \le 2^{k/2}\|f_i\|_{3/2} \ .$$

Putting these together yields

$$I_i(f) \le 2^k\|f_i\|_{3/2}^2 = 2^k(I_i(f))^{4/3} \ ,$$

so either $I_i(f) = 0$ or we can divide by it getting

$$I_i(f) \ge 8^{-k} \ .$$

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Since $\sum_i I_i(f) = \sum_S |S|\hat{f}^2(S) \le k$ we get one final corollary.

Corollary 12 *Let $f : \{\pm1\}^n \to \{\pm1\}$ be a Boolean function of degree at most k. Then the number of influencing variables is at most $k8^k$.*

note: one can actually get a better exponent basis in this bound, but some exponent is necessary (exercise).