

Harmonic Analysis of Boolean Functions, and applications in CS

Lecture 1

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This course covers topics concerning harmonic analysis of *Boolean functions*, and their relations to computer science. We often consider these functions as special cases of functions to \mathbb{R} .

Definition 1 A function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is called a *Boolean function*.

The main application of harmonic analysis is deducing *global properties* from *local* ones:

Example 1 Suppose that a Boolean function f satisfies the following property:

$$\forall x, y \in \{-1, 1\}^n : f(x) \cdot f(y) = f(x \cdot y). \quad (1)$$

Property 1 is **local** in the sense that in order to verify it we only need to consider point triplets. Using harmonic analysis we can easily infer that f satisfies the following **global** property:

$$\exists S \subseteq [n] : f(x) = \prod_{i \in S} x_i. \quad (2)$$

That is, f is the parity function of a subset of the coordinates.

Property 2 can also be deduced without the use of harmonic analysis. However, it is not clear how one can obtain the next robustness property with good parameters without harmonic analysis.

Example 2 (robustness) Suppose that f does not satisfy the above property, but is “close” to satisfying it. Specifically, suppose

$$\Pr[f(x) \cdot f(y) = f(x \cdot y)] \geq 0.9. \quad (3)$$

Using harmonic analysis it can be shown that f is close to a parity function. Namely, one can show that f satisfies a property of the form:

$$\exists S \subseteq [n] : \Pr \left[f(x) = \prod_{i \in S} x_i \right] \geq p \quad (4)$$

for some $p \geq 0.7$. For now, we are not interested in the best possible value of p .

Harmonic analysis turns out to be useful in different fields of theoretic CS, e.g.: property testing, proof checking (PCP), circuit lower bounds, graph threshold phenomena.

Influence

The *influence* of an input variable x_i on a function f is the probability that flipping this variable at a random input x will alter the value of the function f . The influence measures how "energetic" f is; a high $I(f)$ means that f is typically quite sensitive to bit flips, a low $I(f)$ indicates that it is not so sensitive.

Denote $e_i = (1, \dots, 1, \underset{\text{pos } i}{-1}, 1, \dots, 1)$. For $x = (x_1, \dots, x_n)$ and $y = (x_1, \dots, x_n)$ let $x \cdot y = (x_1 \cdot y_1, \dots, x_n \cdot y_n)$.

Definition 2 (influence) Let f be a Boolean function. The influence of the i^{th} coordinate on f is $I_i(f) = \Pr_x[f(x) \neq f(x \cdot e_i)]$. The total influence of f 's coordinates is $I(f) = \sum_{i=1}^n I_i(f)$.

Note that $I(f) = n \cdot \Pr_{x,i}[f(x) \neq f(x \cdot e_i)]$.

Influence is a useful parameter when studying boolean functions, and harmonic analysis allows us to answer influence-related questions.

Example 3 Denote $\text{parity}(x) = \prod_{i \in [n]} x_i$. $I(\text{parity}) = n$.

Exercise 1 Show $I(f) = n$ implies $f = \pm \text{parity}$.

▷ What is the Boolean function with the smallest total influence? Clearly $f(x) \equiv 1$ has total influence 0. To make this question more interesting, we require the function to be *balanced*, i.e. attain -1 and 1 an equal number of times.

Definition 3 (balanced function) f is called *balanced* if $\mathbb{E}_x[f(x)] = 0$.

▷ Now, what is the balanced Boolean function with the smallest influence? One option to consider is the following:

Example 4 $f(x) = x_{17}$ is a balanced function with low influence, $I(f) = 1$. In general, $I(\text{dictatorship}) = 1$ for any function dictatorship that depends only on a single variable.

It turns out that dictatorship has the minimal influence among all balanced functions. Furthermore, we will show that this result is robust. Without the use of harmonic analysis it not clear how this result can be proved, even without the robustness property.

The function suggested in the last example depends only on coordinate.

▷ What if f is balanced, but must depend on all the coordinates?

Denote by S_n the group of all permutations of n elements.

Definition 4 (symmetric function) A Boolean function f is called *symmetric* if $\forall \pi \in S_n, (x_1, \dots, x_n) \in \{-1, 1\}^n$ it holds that $f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$.

Definition 5 A group $G \subseteq S_n$ is called *transitive* if $\forall i, j \exists \pi \in G$ such that $\pi(i) = j$.

Definition 6 (transitive function) A Boolean function f is called *transitive* if there exists a transitive group of permutations G such that $\forall \pi \in G, (x_1, \dots, x_n) \in \{-1, 1\}^n$ it holds that $f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$.

Note that every symmetric function is also transitive (take $G = S_n$).

Example 5 $f(x_1, x_2, x_3, x_4) = (x_1 \wedge x_2) \vee (x_3 \wedge x_4)$ is transitive. E.g., for $i = 1$ and $j = 4$ the permutation $\pi = (4312)$ satisfies $\pi(1) = 4$ and $f(x_1, x_2, x_3, x_4) = f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = f(x_4, x_3, x_1, x_2)$.

Exercise 2 Show that if f is transitive then $\forall i, j$ it holds that $I_i(f) = I_j(f)$.

Let us consider the simplest symmetric function one can think of:

Example 6 The majority function $\text{maj}(x) = \text{sign}(\sum_{i=1}^n x_i)$ is symmetric and balanced for odd n 's. Assume for simplicity that n is odd. It holds:

$$\begin{aligned} I(\text{maj}) &= n \cdot \Pr_{x:i} [\text{maj}(x) \neq \text{maj}(x \cdot e_i)] \\ &\leq n \cdot \Pr_x \left[\sum_{i=1}^n x_i \in \{1, -1\} \right] \\ &\approx n \cdot \frac{1}{\sqrt{n}} \\ &= \sqrt{n} \end{aligned}$$

▷ Are there transitive balanced functions with maximal influence smaller than that of the majority function? Yes! We introduce a function called tribes. The tribes function is transitive, but has a smaller symmetry group.

Example 7 Denote $n = c \cdot k \lg k$, with some constant c . We partition the n coordinates into ck "tribes" of size $\lg k$ each. For a given $x \in \{-1, 1\}^n$, we set $f(x) = 1$ iff there exists a tribe such that all members of the tribe are 1. Formally:

$$f(x) = (x_1 \wedge \dots \wedge x_{\lg k}) \vee (x_{\lg k+1} \wedge \dots \wedge x_{2 \lg k}) \vee \dots \vee (x_{n-\lg k+1} \wedge \dots \wedge x_n).$$

Note that the function is transitive since all tribes are of the same size. The function is also (roughly) balanced: The probability that any given tribe is voting unanimously 1 is $2^{-\lg k} = \frac{1}{k}$. Thus, the probability that no tribe gives a unanimous 1 is $(1 - \frac{1}{k})^{ck} \rightarrow \frac{1}{2}$ by 1Tf TD1 with

We conclude that $I(f) = \theta(\lg n)$.

It turns out that the tribes function has the minimal total influence among all balanced transitive Boolean functions, and majority among all the symmetric ones.

The minimality of the total influence of the tribes function can be deduced from the following result of Friedgut.

Definition 7 (J -junta) f is a J -junta if it depends on at most J coordinates.

Definition 8 ((J, ϵ) -junta) f is a (J, ϵ) -junta if there exists a J -junta g such that

$$\Pr_x[f(x) = g(x)] \geq 1 - \epsilon.$$

Theorem 9 (Friedgut 94') Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$. If $I(f) \leq k$ then for every $\epsilon > 0$, f is $(9^{\frac{k}{\epsilon}}, \epsilon)$ -junta.

Friedgut's Theorem is a kind of "symmetry-breaking" result. Suppose that we try to construct an "almost transitive" function with low influence. We can push the influence lower and lower until $\theta(\log n)$ while keeping the function close to transitive, but as soon as one pushes the influence below this threshold, it has to break symmetry and become essentially a kind of junta.

This phenomenon has analogies in physics, and it is very useful in theoretical computer science. We may see some applications of this phenomenon in this course.