Harmonic Analysis of Boolean Functions, and applications in CS

Lecture 11

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1 Hardness of approximation of E3-LIN-2

W r fr sh from last l ctur th d finition of th uniqu gam probl m with param t r k, UG[k], and of th Uniqu Gam s Conj ctur (UGC).

An instanc I of th probl m compris s of a s t of v rtic s, V, and a s t of dir ct d dg s, E. Each dg $e \in E$ has an associat d w ight, $w(e) > 0$ such that $w(E) = \sum_{e \in E} w(e) = 1$ and an associat d p rmutation $\tau_e \in S_k$. An assignm nt A for I is $A : V \to [k]$, and $val_I(A) = \mathbb{P}_{e=(u,v)\sim E}\left[A(v) = \tau_e(A(u))\right]$.

UGC stat s that for all small nough $\delta, \epsilon > 0$ th r is a k such that distinugishing b tw n an instanc $I \in$ UG[k] that satisfi s $opt(I) \geq 1 - \epsilon$ and an instanc $J \in$ UG[k] for which $opt(J) \leq \delta$ is NP-hard.

Whil hardn ss for UG[k] is only conj ctur d, w can actually prov it for LC[k] — wh r w hav a g n ral function inst ad of a p rmutation. Not that LC[k] is hard with a $(1, \delta)$-gap, which is cl arly impossibl in th cas of UG.

Th goal of this l ctur is to prov that UGC impli s hardn ss of approximation for E3-LIN-2, which was d fin d in th pr vious l ctur .

Theorem 1 *Assuming UGC, for all small enough $\delta, \epsilon > 0$, it is NP-hard to distinguish between instances of E3-LIN-2 which satisfy $opt(I) > 1 - \epsilon$ and instances where $opt(I) < \frac{1}{2} + \delta$.*

W prov Th or m 1 by showing a (polynomial-tim) r duction r[k] from an instanc I in UG[k] to an instanc I' of E3-LIN-2 such that

$$opt(I) > 1 - \epsilon \implies opt(I') > 1 - 2\epsilon \tag{1}$$

and

$$opt(I) < \frac{\delta^3}{32 \log_{(1-2\epsilon)}(\delta/4)} \implies opt(I') < \frac{1}{2} + \delta, \tag{2}$$

and th numb r of quations in I' is bound d by $C(\delta, \epsilon) \cdot |V(I) + E(I)|$.

Giv n th s t of v rtic s in I, $V(I)$, w g n rat a s t of variabl s $V' = \{f_v(x)\}_{\substack{v \in V \\ x \in \{\pm 1\}^k}}$.

Thus $|V'| = 2^k |V|$.

For E' w pick $e = (u, v) \sim E$ and apply th p rmutation-t st with param t r ϵ on f_u, f_v and th p rmutation τ_e, using th random param t rs $x, y \sim \mu^{(k)}_{1/2}, z \sim \mu^{(k)}_{\epsilon}, \eta \sim \mu^{(1)}_{1/2}$.

Explicitly, w hav

$$E' = \left\{ f_u(x) f_u(y) = \eta f_v\left(\eta \tau_e\left(xyz\right)\right) \ : \ x, y, z \in \{\pm 1\}^k, \eta \in \{\pm 1\}, e = (u, v) \in E \right\},$$

wh r th w ight of ach quation is $w\left((u, v)\right) 2^{-3k} (1-\epsilon)^{|\{i : z_i = 1\}|} \epsilon^{|\{i : z_i = -1\}|}$. Notic that $|E'| = 2^{3k+1} |E|$.

1.1 Corr ctn ss of R duction

To compl t th proof of th th or m, it is nough to show that th abov r duction has prop rti s (1) and (2).

First w show (1) (compl t n ss):

Assum $opt(I) > 1 - \epsilon$, and l t A b an assignm nt with $val_I(A) > 1 - \epsilon$. D fin A' for I' by s tting $A'(f_v(x)) = \chi_{A(v)}(x)$. Fixing th assignm nt A', th notation w chos for th variabl s of th E3-LIN-2 instanc allow us to vi w th m as functions of th binary word x. Thus, som what abusing notation, w id ntify $f_v(x)$ with its assignm nt $A'(f_v(x))$.

$$\begin{aligned} val_{I'}(A') &= \mathbb{P}_{e \sim E'}\left[A' \text{ satisfi s } e\right] \\ &\geq \mathbb{P}_{e \sim E}\left[A \text{ satisfi s } e\right] \mathbb{P}_{x,y,z,\eta}\left[f_u(x) f_u(y) = \eta f_v\left(\eta \tau_e\left(xyz\right)\right) \ | \ A \text{ satisfi s } e\right] \\ &\geq val_I(A)(1-\epsilon) \geq (1-\epsilon)^2 > 1 - 2\epsilon, \end{aligned}$$

wh r conditioning on A satisfying e gav us that $f_v = \chi_{A(v)} = \chi_{\tau_e(A(u))} = f_u$, and thus w may us th $1 - \epsilon$ compl t n ss of th p rmutation t st.

To prov (2), w show th contrapositiv , nam ly, that if $r[k](I) = I'$, and $\exists A'$ with $val_{I'}(A') \geq \frac{1}{2} + \delta$ th n $opt(I) \geq \frac{\delta^3}{32 \log_{(1-2\epsilon)}(\delta/4)}$. Th assignm nt A' d fin s a function $f_v(\cdot)$ for ach $v \in V$. To us soundn ss of th p rmutation t st, w n d to show th r ar nough dg s $(u, v) \in E$ for which $f_v(x), f_u(x)$ satisfy th ir quations in I' with good probability on a random x. This will b a cons qu nc of our assumption on $val_{I'}(A')$ which low r bounds th w ight of th $|E'|$ quations that ar satisfi d. Using soundn ss, w will randomly d cod $f_v(\cdot)$ for ach v rt x in V and prov that th av rag valu of our random assignm nt to I is low r bound d by th function of δ app aring in 2. Sinc th r is a d t rministic choic of A for which $val(I)$ is at l ast th av rag , this will prov what w want.

For ach $v \in V$ choos D^v at random from ith r $D_1(f_v)$ or $D_2(f_v)$ (th p rmutation t st d cod rs for th first and s cond word) to obtain a word χ_i, and l t A assign i to v (if w g t $\perp$ w assign an arbitrary lab l).

W us th following l mma to prov that $\mathbb{E}\left[val_I(A)\right]$ is larg .

Lemma 2 *Let X be an r.v. satisfying $0 \leq X \leq 1$, then*

$$\mathbb{P}\left[X \geq \alpha\right] \geq \frac{\mathbb{E}\left[X\right] - \alpha}{1 - \alpha} \tag{3}$$

Proof

$$1 \cdot \mathbb{P}\left[X \geq \alpha\right] + \alpha(1 - \mathbb{P}\left[X > \alpha\right]) \geq \mathbb{E}\left[X\right]$$

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For $e = (u, v) \in E$, l t $P_e = \mathbb{P}_{x,y,z,\eta}$ [p rmutation t st[ϵ] acc pts on f_u, f_v, τ_e with A']. Th law of total probability giv s

$$\frac{1}{2} + \delta < val_{I'}(A') = \mathbb{P}_{e \sim E}\left[A' \text{ sat. } e\right] = \mathbb{E}_{e=(u,v)\sim E}\left[P_e\right].$$

Thus using 3,

$$\mathbb{P}_{e=(u,v)\sim E}\left[P_e \geq \frac{1}{2} + \frac{\delta}{2}\right] \geq \frac{\delta/2}{1 - (1/2 + \delta/2)} > \delta.$$

L t $\mathcal{E} = \left\{e \in E \ : \ P_e \geq \frac{1}{2} + \frac{\delta}{2}\right\}$ b th subs t of dg s for which th t st pass s with good probability. Th abov shows th probabilistic w ight of dg s in $\mathcal{E}$, $w(\mathcal{E}) = \sum_{e \in \mathcal{E}} w(e)$ is at l ast δ. For $e = (u, v) \in E$, l t $\pi(e) = \mathbb{P}_{\chi_u \sim D_1(f_u), \chi_v \sim D_2(f_v)}\left[\chi_u(x) = \chi_v(\tau_e(x))\right]$ and r call that th p rmutation t st has satisfaction rat $s(\alpha) = \frac{\alpha^2}{2\log_{(1-2\epsilon)}(\alpha/2)}$ for soundn ss of $\frac{1}{2} + \alpha$. Thus for $e \in \mathcal{E}$, $\pi(e) \geq s(\frac{\delta}{2})$.

Finally, w low r bound $\mathbb{E}\left[val_I(A)\right]$ (th random d cod r choic is av rag d out) as follows

$$\begin{aligned}
\mathbb{E}\left[val_I(A)\right] &= \mathbb{E}_{\substack{e=(u,v)\sim E \\ D^u, D^v \in_R \{D_1, D_2\}}}\left[\mathbf{1}_{\{A \text{ satisfi s } e\}}\right] \\
&\geq \sum_{e=(u,v)\in\mathcal{E}} w(e)\mathbb{E}\left[\mathbf{1}_{\{D^u = D_1, D^v = D_2\}}\mathbf{1}_{\pi(e)} | e \in \mathcal{E}\right] \\
&\geq w(\mathcal{E}) \min_{e \in \mathcal{E}}\left\{\mathbb{E}\left[\mathbf{1}_{\{D^u = D_1, D^v = D_2\}}\mathbf{1}_{\pi(e)}\right]\right\} \geq \frac{\delta}{4} s(\frac{\delta}{2}) \\
&= \frac{\delta^3}{32\log_{(1-2\epsilon)}(\delta/4)}.
\end{aligned}$$

and this is what w want d.