Harmonic Analysis of Boolean Functions in Computer Science

Assignment no. 5

Date due: 6/23/2008

1. For any (small enough) positive constants ϵ, δ, prove that when the 'function test' shown in class (also known as the *projection test*) is run with parameter ϵ, it has soundness $\frac{1}{2} + \delta$.

2. Let $f : \{-1,1\}^n \to \mathbb{R}$ be a function of degree k. Show that for some global constants $c, d > 0$,
$$\Pr_{x \sim \mu_{1/2}^{(n)}} [|f(x)| > t] \leq c \cdot \exp(-d \cdot t^{2/k}).$$

3. (a) Let $f : \{-1,1\} \to \{0,1\}$ be a function. Taking $t \doteq \Pr_{x \sim \mu_{1/2}^{(n)}} [f(x) = 1]$, assume that $t < 1/2$ and show that for some global constant $c > 0$,
$$I(f) \geq c \cdot t \cdot \log(1/t). \qquad (1)$$
   (b) Let $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function, and let $v \doteq \mathbb{V}_{x \sim \mu_{1/2}^{(n)}} [f(x)]$ (recall that $\mathbb{V}$ denotes the variance). Show that for some global constant $c > 0$,
$$I(f) \geq c \cdot v \cdot \log(1/v).$$
   (c) Show that (1) is tight up to the value of the constant c.

4. (a) Let f be a boolean function that has a decision tree of height k (if you do not know what a decision tree is, first define it as you see fit and then prove this section). Show that the degree of f is at most k.
   (b) Show that for every k there exists a boolean function of degree k which depends non-trivially on at least $2^k - 1$ coordinates.

5. For a function $g : \{-1,1\}^n \to [0, \infty)$, define $\mathbf{Ent}(g) \doteq \mathbb{E}_x \left[g^2(x) \ln(g^2(x))\right] - \mathbb{E}_x [g(x)] \ln(\mathbb{E}_x [g(x)])$ (where we take $0 \cdot \ln 0 = 0$). Setting $\psi(\epsilon) \doteq \|f\|_{2-2\epsilon}^2$, show that the derivative of ψ at $\epsilon = 0$ is $-\mathbf{Ent}(f)$.

6. Let $f : \{-1,1\}^n \to \mathbb{R}$ be any function.
   (a) For $\epsilon \geq 0$ define $\varphi(\epsilon) \doteq \|T_{\sqrt{1-2\epsilon}}(f)\|_2^2$. Compute the derivative of φ at $\epsilon = 0$.
   (b) Prove that
$$\mathbf{Ent}(f) \leq 2I(f). \qquad (2)$$
   (c) Use (2) to reprove (1), perhaps with a better constant. If you already used (2) in section (3a), solve section (3a) again without using (or reproving) (2).

7. **(Bonus 15pts)** Show that there exist global constants $c, d > 0$ with the following property. For any k and any function $f : \{-1,1\}^n \to \mathbb{R}$ of degree k, letting $\eta = \mathbb{E}_x [f(x)]$ we have
$$\Pr_x [f(x) \geq \eta] > c \cdot \exp(-dk).$$