Harmonic Analysis of Boolean Functions in Computer Science

Assignment no. 4

Date due: 5/19/2008

1. (a) Let $f : \{-1,1\}^n \to \mathbb{R}$ be a transitive function. Show that $I_i(f) = I_j(f)$ for all $i, j \in [n]$.
   (b) Let n be odd. Show that the majority function has the highest weight on the first level (linear characters) out of all Boolean valued functions.

2. Let $C = C_1 = C_2 = \{\chi_S\}_{S \subseteq [n]}$, and let $\mathcal{R} = \{R_T\}_{T \subseteq [n]}$ be a constraint family where for every T, $R_T = \{f, g \in C : \ f = \chi_T g\}$. Show a three query test for the codes C_1, C_2 and the constraint family $\mathcal{R}$ that has completeness 1 and soundness $\frac{1}{2} + \delta$ for every constant $\delta > 0$.

3. (a) Find a reasonable generalization of a q-query constraint test to the three-way case, namely the case where three code families C_1, C_2, C_3 are given and where $\mathcal{R} = \{R_\lambda\}_{\lambda \in \Lambda}$ is a set of relations $R_\lambda \subseteq C_1 \times C_2 \times C_3$. Also generalize the notions of completeness and soundness for this case.
   (b) Find a three query three-way constraint tester for the codes $C = C_1 = C_2 = C_3 = \{\chi_S\}_{S \subseteq [n]}$ and for the constraint family $\mathcal{R} = \{R_{T_1,T_2}\}_{T_1,T_2 \subseteq [n]}$, where

   $$R_{T_1,T_2} = \{f, g, h \in C : \ g = \chi_{T_1} \cdot f \ \text{ and } \ h = \chi_{T_2} \cdot f\}.$$

   Your test should have completeness 1. In addition, if for any f, g, h, T_1, T_2 and $\delta > 0$ your test succeeds with probability at least $1 - \delta$, it must hold that for some $u, v, w \in \{\pm 1\}$,

   $$\max \{\|f - u \cdot \chi_S\|_2^2, \|g - v \cdot \chi_{T_1}\chi_S\|_2^2, \|h - w \cdot \chi_{T_2}\chi_S\|_2^2\} \leq O(\delta).$$

   (c) Prove that the test you constructed in the previous section has soundness $1/2 + \delta$ for any constant $\delta > 0$ (or alternatively, solve the previous section again with a new test, and then prove this additional property..).

4. Let $f : \{-1,1\}^n \to \{-1,1\}$ be a function, let $x \in \{-1,1\}^n$ and $u \in \{\pm 1\}$ be uniformly selected, and for some small positive parameter $\epsilon < 1/2$, let $z \in \{-1,1\}^n$ be randomly chosen by independently setting

   $$z_i = \begin{cases} -1 & \text{with probability } \epsilon \\ 1 & \text{with probability } 1 - \epsilon. \end{cases}$$

   Let T be the two-query test which accepts iff $f(x) = u \cdot f(uzx)$. Show that for every (small enough) $\epsilon > 0$ there is a number $\delta(\epsilon) > 0$ such that the following hold:

   - If $f = \chi_i$ or $f = -\chi_i$ for some i, then the test accepts with probability at least $1 - \epsilon$.
   - If the test accepts with probability at least $1 - \epsilon - \delta(\epsilon)$ then f must be a non-constant Boolean dictatorship g such that $\|f - g\|_2^2 \leq 0.1$.

5. (a) Extend the test from question (4) to a permutation test over the long-code, and prove that it has completeness $1 - \epsilon$ and soundness $\delta(\epsilon)$ for some $\delta(\epsilon) > 0$.
   (b) Show that for every (small enough) $\epsilon > 0$ there exists a number $\delta(\epsilon) > 0$ with the following property: assuming the Unique Games conjecture, it is NP-hard to distinguish between instances of Max-E2-LIN-2 with optimum at least $1 - \epsilon$ and instances where the optimum is at most $1 - \epsilon - \delta(\epsilon)$.