Harmonic Analysis of Boolean Functions in Computer Science

Assignment no. 3

Date due: 4/28/2008

1. (a) Let $f = \sum_{i=1}^{n} a_i \chi_i$ be a homogeneous linear function (homogeneous means that it only has degree-one terms, namely the coefficient of the constant character is zero). Write $\|f\|_4$ in terms of the Fourier coefficients of f.

   (b) Use your formula to prove that there exists a global constant c such that for every homogeneous linear function f, $\|f\|_4 \leq c \cdot \|f\|_2$.

   (c) Show that the inequality from (1b) does not hold for all functions.

2. Let Ω be a probability space. The indicator $\mathbf{1}_A : \Omega \to \mathbb{R}$ of a set $A \subseteq \Omega$ is defined to be 1 on elements of A and zero otherwise. Recall that $X : \Omega \to \mathbb{R}$ is a *discrete random variable* over Ω if and only if it can be written in the form $X = \sum_{i \in \Lambda} \alpha_i \cdot \mathbf{1}_{A_i}$, where the sets $\{A_i\}_{i \in \Lambda}$ are measurable and disjoint.

   Let X be a discrete non-negative random variable.

   (a) Prove that
   $$\mathbb{E}[X] = \int_{t=0}^{\infty} \Pr[X > t]\, dt.$$

   (b) Show that for every parameter $p \geq 1$,
   $$\mathbb{E}[X^p] = p \cdot \int_{t=0}^{\infty} t^{p-1} \cdot \Pr[X > t]\, dt.$$

3. Prove that there exists a global constant C such that for every homogeneous linear function $f : \{-1,1\}^n \to \mathbb{R}$ and for every even integer $p \geq 2$,
   $$\|f\|_p \leq C\sqrt{p} \cdot \|f\|_2.$$

4. Let f be a function satisfying $\|f\|_p \leq \sqrt{p}$ for all $p \geq 1$. Show that for every $t > \sqrt{e}$,
   $$\Pr[|f| > t] \leq \exp\left(-\frac{t^2}{2e}\right).$$

5. (a) Let $p \geq 1$ and let $\{a_i\}_{i=1}^{n}$ be any real coefficients. Show that for every $l \leq n$,
   $$\left\|\sum_{i=1}^{l} a_i \chi_i\right\|_p \leq \left\|\sum_{i=1}^{n} a_i \chi_i\right\|_p.$$

   (b) Show that in general, zeroing the coefficient of a character may actually increase the L_p norm of a function.

   (c) Show that if for all i, $|b_i| \geq |a_i|$ then
   $$\left\|\sum_{i=1}^{n} a_i \chi_i\right\|_p \leq \left\|\sum_{i=1}^{n} b_i \chi_i\right\|_p.$$