Harmonic Analysis of Boolean Functions in Computer Science

Assignment no. 1

Date due: 3/10/2007

No Fourier. In this exercise set please do not use Fourier analysis in your answers.

1. Let M_n denote the majority function on n variables. Compute
$$\lim_{n\to\infty} \frac{I(M_n)}{\sqrt{n}}$$
(use Stirling's approximation formula if necessary).

2. Prove that there exists a constant c such that for every constant $b > 0$ and for every large enough t the following holds: there is a number $m(t)$ such if we take $n = t \cdot m(t)$ and let $f\colon \{-1,1\}^n \to \{-1,1\}$ be the tribes function with $m(t)$ tribes of size t each, then $|\mathbb{E}_x[f(x)]| < b$ and $I(f) \le c \cdot \log n$.

3. Let $f\colon \{-1,1\}^n \to \{-1,1\}$ satisfy $I(f) \ge n - \epsilon$. Show that either $\Pr_x[f(x) \ne \prod_{i=1}^n x_i] \ge 1 - \epsilon/2$ or $\Pr_x[f(x) \ne -\prod_{i=1}^n x_i] \ge 1 - \epsilon/2$.

4. (a) Let X_1 and X_2 be independent random variables, and let $f(X_1, X_2)$ be a function. Suppose that $\mathbb{V}_{X_2}[\mathbb{E}_{X_1}[f(X_1,X_2)]] = \mathbb{E}_{X_1}[\mathbb{V}_{X_2}[f(X_1,X_2)]]$, and prove that there exist functions $g(X_1)$ and $h(X_2)$ such that $f(X_1, X_2) = g(X_1) + h(X_2)$.

   (b) Find a necessary and sufficient condition for a function $f : \{-1,1\}^n \to \mathbb{R}$ to satisfy $\mathbb{V}_x[f(x)] = I(f)$.

   (c) Let $f : \{-1,1\}^n \to \{-1,1\}$ be a balanced function such that $I(f) = 1$. Show that f is a dictatorship.

5. Let's generalize the notion of influence from single coordinates to sets of coordinates. Fix $f\colon \{-1,1\}^n \to \mathbb{R}$. For a set of $S \subseteq [n]$ of coordinates we define the variation of f on S by
$$\mathsf{Vr}_S(f) \doteq \mathop{\mathbb{E}}_{x\setminus S}\left[\mathop{\mathbb{V}}_{x\cap S}[f(x)]\right].$$
Show that for every $S, T \subseteq [n]$,
$$\mathsf{Vr}_{S\cup T}(f) \le \mathsf{Vr}_S(f) + \mathsf{Vr}_T(f).$$