Solution no. 7

1. Let $x, y \in U$ be chosen randomly and independently. Then
$$\Pr[h(x) = h(y)] = \sum_{i=1}^{m} \Pr[h(x) = h(y) = i] = \sum_{i=1}^{m} \Pr[h(x) = i] \Pr[h(y) = i] = \frac{m}{m^2} = \frac{1}{m}$$

2. The expected running time is linear in n. This is easy to obtain by induction, noting that the recursive formula for the complexity is given by
$$T(n) = n + \frac{1}{n} \sum_{i=0}^{n-1} T(i)$$

3. Insert all elements to a hash-table which counts the multiplicity of each inserted element. Then sort the elements in the hash-table, remembering each element's multiplicity, and finally print the sorted elements, printing each element a number of times according to its multiplicity. Since there are $\frac{n}{\log n}$ elements, it is easy to verify that the sorting takes linear time in n. Everything else takes expected linear time as well.

4.
$$\mathbb{E}[Z] = \sum_{i=0}^{\infty} i \Pr[Z = i] \geq \sum_{i=k+1}^{\infty} i \Pr[Z = i] \geq (k+1) \sum_{i=k+1}^{\infty} \Pr[Z = i] = (k+1) \Pr[Z > k]$$
The desired inequality now follows directly.

5. Fix $x_0, y_0 \in U$ to be two distinct elements, and let h be a random function from the collection $\mathcal{H}$. Then by the definition of a universal collection, $\Pr[h(x_0) = h(y_0)] \leq 1/m$. If, however, $x_0 = y_0$, the latter probability equals 1. It follows, using the law of total probability, that for randomly and independently chosen $x, y \in U$,
$$\Pr[h(x) = h(y)] = \Pr[x \neq y] \cdot \Pr[h(x) = h(y) \mid x \neq y] + \Pr[x = y] \leq \Pr[x \neq y] \cdot \frac{1}{m} + \frac{1}{|U|} < \frac{1}{m-1}$$
Now assume for the sake of contradiction that one of the values in $\{0, \ldots, m-1\}$ is never attained by any function in $\mathcal{H}$, and let us show that $\Pr[h(x) = h(y)] \geq 1/(m-1)$, thus achieving a contradiction. Without loss of generality we can assume that this value is 0. For $j = 1, \ldots, m-1$, let p_j denote the probability that $h(x) = j$ (where h and x are as defined above). Then, with the same notation as above, it follows that
$$\Pr[h(x) = h(y)] = \sum_{j=1}^{m-1} {p_i}^2 \geq \frac{1}{m-1}$$
The last inequality above is left as an exercise to the reader. It can be proven using the Cauchy-Schwartz inequality and the fact that $\sum_{j=1}^{m-1} p_i = 1$.