Advanced Algorithms - Recitation	January 15, 2006
Lecture 14	
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1 Demands Multicommodity Flow

Problem definition: We are given a weighted graph $G = (V, E)$, where each edge e has a capacity of $c(e)$. We have k different goods; each good i travels the graph from s_i to t_i, and has demand $dem(i)$.
Our goal: to find and maximize f so that each good i has a flow of $f * dem(i)$ units.

For any cut S, we define $c(S) = \sum_{e\in(S,\overline{S})} c(e)$, and $dem(S)$ is the summation of $dem(i)$ for every i whose flow the cut S disrupts (i.e. flow is not possible from s_i to t_i after the cut). By these definitions, we immediately find:

$$f \leq \frac{c(S)}{dem(S)}$$

We will refer to $\frac{c(S)}{dem(S)}$ as the sparsity of cut S.

Last lesson, we showed that the optimal (fractional) answer can be viewed as a metric over the vertices V. In the LP problem, d_e is a variable representing whether or not e is in the cut; after relaxation $d_e \in [0, 1]$. For each two vertices $u, v \in V$, $d(u, v)$ is the sum of all d_es on the shortest path from u to v.

Definition 1 *A function $y : 2^V \to \mathbb{R}$ is said to be a β-*approximate-cut-packing *if for every $e \in E$, where d_e is e's distance in the metric,*

$$\frac{d_e}{\beta} \leq \sum_{S|e\in S} y(S) \leq d_e$$

Theorem 2 *Let (V, d) be the metric determined by LP, as seen last lesson. Let y be a β-approximate-cut-packing for (V, d). Of all the cuts S where $y(S) \neq 0$, let S' be the cut with minimal sparsity ($\frac{c(S)}{dem(S)}$). Then:*

$$\beta OPT \leq \frac{c(S')}{dem(S')}$$

Lemma 3 *Let $\sigma : V \rightarrow \mathbb{R}^m$ be a mapping. Then there exists a cut-packing y so the weight each edge e 'feels' ($\sum_{S|e\in S} y(S)$) is e's length in the metric l_1 under σ.*
In other words, for $e = (u, v)$:

$$\sum_{S|e\in S} y(S) = ||\sigma(u) - \sigma(v)||_{l_1}$$

Theorem 4 *There exists an $O(logn)$ approximation to the SPARSEST-CUT problem.*

Proof: We will find a β-approximation-cut-packing (β-approx-CP) for $\beta = O(logn)$; this suffices to prove the theorem.
We will embed our metric (V, d) in l_1, using mapping σ. By Bourgain, we have a distortion of $O(logn)$, which we will denote β. So for every e, we find for $e = (u, v)$

$$\frac{d_e}{\beta} \leq ||\sigma(u) - \sigma(v)||_{l_1} \leq d_e$$

Using the lemma, there exists a cut-packing y so that $\sum_{S|e\in S} y(S) = ||\sigma(u) - \sigma(v)||_{l_1}$, and so we find

$$\frac{d_e}{\beta} \leq \sum_{S|e\in S} y(S) \leq d_e$$

which is precisely a β-approx-CP, as required. ■

Proof: We prove the lemma for $m = 1$.
We have a mapping σ from our n vertices to $u_1 \leq u_2 \leq ... \leq u_n \in \mathbb{R}$ (in order, without loss of generality). We define:
$y(\{v_1\}) = u_2 - u_1$
$y(\{v_1, v_2\}) = u_3 - u_2$
...
$y(\{v_1, ..., v_k\}) = u_{k+1} - u_k$
For any set $K \subseteq V$ not of the form $\{v_1, v_2, ..., v_k\}$, we'll define $y(K) = 0$.
For any cut S where $(u_1, u_n) \in S$, we find:

$$\sum_{S|e\in S} y(S) = \sum_{S|v_1\in S, v_n\notin S} y(S) = \sum_{1\leq k<n} y(\{v_1, v_2, ..., v_k\}) = u_n - u_{n-1} + u_{n-1} - u_{n-2} + ... + u_2 - u_1 = u_n - u_1$$

And so we have found that (u_1, u_n) feels $u_n - u_1 = ||v_n - v_1||_{l_1}$ Similarly, for any $(u_i, u_j) \in S$, $i \leq j$, we find

$$\sum_{S|e\in S} y(S) = u_j - u_i = ||v_j - v_i||_{l_1}$$

since the first cut of the $\{v_1, ..., v_k\}$ form that contains v_i is $\{v_1, ..., v_i\}$, and the last that does *not* contain v_j is $\{v_1, ...v_{j-1}\}$.
Thus, we have proved the lemma for $m = 1$.

It is simple to generalize to m dimensions - simply project onto each dimension separately, and repeat the process to find the distance along that dimension. l_1 distance is found by summation of the distances found along each dimension. ■