Advanced Algorithms - Lecture	December 8, 2005

Lecture 7

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1 LP Duality

Let us examine an LP equation system, as follows:

- Minimize $f(x) = x_1 + 2x_2 + 4x_3$ so that:

$$\begin{aligned} &\text{(I)} && x_1 + x_2 + 2x_3 = 5 \\ &\text{(II)} && 2x_1 + x_2 + 3x_3 = 8 \\ &\text{(III)} && \forall i: \quad x_i \geq 0 \end{aligned}$$

We will denote the optimal solution x^*, and its value P^*.
We can immediately see that

$$P^* \geq f(x) \geq (I) = 5$$

Similarly,

$$P^* \geq f(x) \geq x_1 + 2x_2 + 3x_3 = 3(I) - (II) = 7$$

For $x = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$, we find that indeed $f(x) = 7 \leq P^* \Rightarrow P^* = 7$. In this manner, it is easy to find upper and lower bounds from *within* the LP system itself.

In order to find the *highest possible* low bound, we will search for the linear combination of equations (I) and (II) which will give us the largest result while still remaining under the value of $f(x)$. In other words, we wish to maximize variables y_1 and y_2 so that:

$$y_1(I) + y_2(II) \leq f(x) \leq P^*$$

$$y_1(I) + y_2(II) = 5y_1 + 8y_2 = y_1(x_1 + x_2 + x_3) + y_2(2x_1 + x_2 + 3x_3) =$$

$$= (y_1 + 2y_2)x_1 + (y_1 + y_2)x_2 + (2y_1 + 3y_2)x_3 \leq f(x) = x_1 + 2x_2 + 4x_3$$

So we have reached a new LP system:

- Maximize $g(y) = 5y_1 + 8y_2$ so that:

$$\begin{aligned} &(\text{I'})\ \ y_1 + 2y_2 \le 1 \\ &(\text{II'})\ \ y_1 + y_2 \le 2 \\ &(\text{III'})\ \ 2y_1 + 3y_2 \le 4 \end{aligned}$$

We denote the optimal solution to be D^*.

Remark Note that in the new LP system, we do *not* demand that $y_i \ge 0$.

Definition 1 *We call the original system the* Primal system*, and the new system the* Dual system*. More precise definitions of these terms will be given shortly.*

Similarly, we can return from the new LP system to the old one. We will want to find the linear combination of equations (I'), (II'), (III') which gives us the smallest result while still remaining higher than $g(y)$. In other words, minimize x_1, x_2, x_3 so that:

$$x_1(I') + x_2(II') + x_3(III') \ge g(y) \ge D^*$$

$$x_1 + 2x_2 + 4x_3 \ge (y_1 + 2y_2)x_1 + (y_1 + y_2)x_2 + (2y_1 + 3y_2)x_3 =$$

$$(x_1 + x_2 + 2x_3)y_1 + (2x_1 + x_2 + 3x_3)y_2 =$$

$$\underbrace{5y_1}_{\text{for } x_1+x_2+2x_3=5} + \underbrace{8y_2}_{\text{for } 2x_1+x_2+3x_3=8}$$

And so we will seek minimal x_1, x_2, x_3 fulfilling those requirements, and we will also require $x_i \ge 0$, or the inequalities' direction will be flipped. All in all, we have precisely recreated the original LP system.

Definition 2 *Given a* Primal system *P, defined with the matrix A and the vectors b and c, we can define P's* Dual system *D as follows:*

Primal P	*Dual* D
$min\ c^T x$	$max\ b^T y$
$Ax = b$	$A^T y \le c$
$x \ge 0$	

Claim 3 *The Dual system of the standard representation of D is equivalent to P.*

Theorem 4 *(The Weak Duality Theorem)*
If x is a solution for P, and y is a solution for D, then $c^T x \geq b^T y$.

Proof: $c^T x \geq (A^T y)^T x = y_T Ax = y^T b$
■

Theorem 5 *(The Strong Duality Theorem)*
If x^* is an optimal solution for P, then there exists an optimal solution y^* for D, and $c^T x^* = b^T y^*$.

Observation 6 *Precisely one of the following situations will take place:*

1. *P is feasible and is not bounded $\Rightarrow$ D is not feasible*
2. *D is feasible and is not bounded $\Rightarrow$ P is not feasible*
3. *P is feasible and bounded $\Leftrightarrow$ D is feasible and bounded.*
4. *Neither P nor D is feasible.*

Let us look at a dual system:

Primal P	Dual D
min $c^T x$ $Ax = b$ $x \geq 0$	max $b^T y$ $A^T y \leq c$

Thus far, for an algorithm ALG and input I, we would find a solution to the LP, round the solution, and find α so that

$$ALG(I) \leq \alpha(C^T x^*) = \alpha OPT_F(I) \leq \alpha OPT(I)$$

Now it is enough to prove that $ALG(I) \leq \alpha(b^T y)$, and then

$$ALG(I) \leq \alpha(b^T y) \quad \underbrace{\leq}_{\text{(weak duality)}} \quad \alpha(c^T x^*) \leq \alpha OPT(I)$$

2 Complementary Slackness

Theorem 7 *Let x, y be solutions for P, D respectively.*
x and y are both *optimal $\Leftrightarrow$ iff for any $i \leq j \leq m$, either $x_j = 0$ or $\sum_i a_{ji} y_i = c_j$*

Proof: $c^Tx - b^Ty = c^Tx - (Ax)^Ty = x^Tc - x^TA^Ty = x^T(c - A^Ty)$
if x, y are optimal, then we've found $c^Tx = b^Ty$:

$$0 = c^Tx - b^Ty = x^T(c - A^Ty = \sum_j x_j(c_j - \sum_i a_{ji}y_i)$$

Both x_j and $(c_j - \sum_i a_{ji}y_i)$ are always ≥ 0, and so the entire expression $= 0$ iff for every j, $x_j = 0$ or $(c_j - \sum_i a_{ji}y_i) = 0$, as required.■

Let us look at a dual system, this time with the system P in the canonical representation:

Primal P	Dual D
min c^Tx	max b^Ty
$Ax \geq b$	$A^Ty \leq c$
$x \geq 0$	$y \geq 0$

Theorem 8 *For P in the canonical representation, let x, y be solutions for P, D, respectively. Then x and y are optimal iff:*

1. $\forall j : x_j = 0$ *or* $\sum_i a_{ji}y_i = c_j$
2. $\forall i : y_i = 0$ *or* $\sum_j a_{ji}x_j = b_i$

3 Expansion of Complementary Slackness

Let P be an LP system in canonical form.

Theorem 9 *If x, y are solutions to P, D, respectively, and* $\alpha, \beta \geq 1$ *fulfill the following conditions:*

1. $\forall j : x_j = 0$ *or* $\sum_i a_{ji}y_i \geq \frac{c_j}{\alpha}$
2. $\forall i : y_i = 0$ *or* $\sum_j a_{ji}x_j \leq \beta b_i$

Then: $c^Tx \leq \alpha\beta b^Ty$.

Corollary 10 *Under the conditions of the theorem,*

$$c^Tx \leq \alpha\beta b^Ty \leq \alpha\beta c^Tx^* \leq \alpha\beta OPT$$

Proof:

$$c^Tx = \sum_j c_jx_j \leq \alpha \sum_j (\sum_i a_{ji}y_i)x_j = \alpha \sum_i (\sum_j a_{ji}x_j)y_i \leq \alpha\beta \sum_i b_iy_i$$

■