Advanced Algorithms - Lecture	January 19, 2006
Lecture 1	
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1 Metric Embeddings - The Bourgain Theorem

We shall start this lecture by presenting Bourgain theorem:

Theorem 1 *It is possible to embed every metric space $M = (V, d)$ with n points in l_2 with a distortion of $o(\log n)$.*

We shall require the use of what is known as the Frechet embedding. We will randomly create t sets, $A_i \subseteq V$ $(1 \leq i \leq t)$. We shall then define t functions, $f_i : V \to R$ such that $f_i(u) = d(u, A_i) = min_{x \in A_i} d(u, x)$. The embedding will then be defined by $f : V \to R^t$, such that $f(u) = (f_1(u), ..., f_t(u))$.

$$|f_i(u) - f_i(v)| = |d(u, A_i) - d(v, A_i)| \leq |d(u, X) - d(v, x)| \leq d(u, v)$$

Therefore,

$$||f(u) - f(v)|| \leq td(u, v)$$

We randomly pick the set A_i by picking every $v \in V$ with probability $\frac{1}{2^i}$. We would like our f_i's to fulfil the following requirement: $\frac{d(u,v)}{24} \leq ||f(u) - f(v)||$. If this will indeed hold then the expansion would be at most $O(\log n)$ and the contraction is $\frac{1}{24}$ and so the distribution is $O(\log n)$.

We will first prove the Bourgain theorem for l_1. Assume that there are t real values, $\Delta_1, ..., \Delta_t$, $E(|f_i(i) - f_i(v)) \geq \frac{\Sigma_i \Delta_1}{6}$ and $\Sigma_i \geq \frac{d(u,v)}{6}$. If so, then $||f(u) - f(v)|| = \Sigma_i E[|f_i(u) - f_i(v)|] \geq \Sigma_i \frac{\Delta_i}{6} \geq \frac{d(u,v}{24}$.

Fix $u, v \in V$. Define $B(x, r) = \{y \in V | d(x, y) \leq r\}$ and $B^0(x, r) = \{y \in V | d(x, y) < r\}$. Let $r_i = min_d\{|B(u, d)| \geq 2^i \ |B(v, d) \geq 2^i\}$ and $\Delta_i = r_i - r_{i-1}$. W.l.o.g. $|B(v, r)| \geq 2^i$ and $|B^0(u, r) < 2^i|$. Consider the expression $d(u, A_i) - d(v, A_i)$. Look at the case in which (*) in $B(u, r_i)$ there is no point from A_i and (**) in $B(v, r_{i-1})$ there is a point from A_i. In this case $d(u, A_i) - d(v, A_i) \geq max\{r_i - r_{i-1}, \frac{d(u,v)}{24}\}$. The probability that this case happens is constant because (*) happens with probability $(1-\frac{1}{2^i})^{2^i} \geq \frac{1}{e}$ and (**) happens with probability $\frac{1}{2}$.

We now have that:

$$E[d(u,A_i)-d(v,A_i)] \geq \frac{r_i-r_{i-1}}{6} \geq \frac{1}{6}\frac{d(u,v)}{4} = \frac{d(u,v)}{24}$$

Now we shall randomly pick, for every i, s independent sets, $A_i^1,...,A_i^s$, as before. For every $1 \leq j \leq s$ and $1 \leq i \leq t$let $f_i^j(u) = d(u,A_i^j)$. The number of coordinates is now $st = o(\log^2 n)$. We define the function f such that the (i,j)'th coordinate of $f(u)$ is $f_{ij}(u) = \frac{1}{st}f_i^j(u)$. It is easy to verify that, as before, $||f(u)-f(v)|| \leq d(u,v)$.

$||f(u)-f(v)|| = \frac{1}{s}\Sigma_j(\frac{1}{t}\Sigma_i|f_i^j(u)-f_i^j(v)|)$. Let X_j be the random variable that gets the value of $\frac{1}{t}\Sigma_i|f_i^j(u)-f_i^j(v)|$. We already know that $\frac{d(u,v)}{24t} \leq E[X_j]$. Let $Y = \frac{1}{s}\Sigma_j X_j$.

As a conclusion from they Chernoff bounds it can be shown that $Pr[Y < (1-\delta)E[X]] < e^{-frac\delta^2 s2}$. That means that for $s = 32\ln n$ and $\delta = \frac{1}{2}$ we have that $Pr[Y < \frac{1}{2}\frac{d(u,v)}{24t}] < e^{-4\ln n} < \frac{1}{2n^2}$.

Since there are n points, the probability that we have "failed" for any of the pairs is at most $n^2\frac{1}{2n^2} = \frac{1}{2}$. This finishes the proof of the theorem for l_1.

In order to handle the l_2 case the normalization will be in $\frac{1}{\sqrt{st}}$. We will then have that:

$$||f(u)-f(v)||_2^2 = \frac{1}{st}(\Sigma_{i,j}|f_i^j(u)-f_i^j(v)|^2) \leq d(u,v)^2$$

$$||f(u)-f(v)|| = \frac{\Sigma|f_i^j(u)-f_i^j(v)|^2}{st} = \frac{1}{s}(\frac{\Sigma|f_i^j(u)-f_i^j(v)|^2}{t})$$

We shall now show that $E[\frac{1}{t}\Sigma|f_i^j(u)-f_i^j(v)|^2] \geq (\frac{d(u,v)}{24t})^2$. To do this we only need to show that $\frac{1}{t}\Sigma|f_i^j(u)-f_i^j(v)|^2 \geq (\frac{1}{t}\Sigma|f_i^j(u)-f_i^j(v)|)^2$. The last inequality is derived form the Cauchy Schwartz inequality.