CS294-1 On-line Computation & Network Algorithms **Spring 1997**

Lecture 22: April 22

Lecturer: Yair Bartal *Scribe: Andrew Swan*

This and the next lecture together cover the work presented in [BBBT]. The ultimate goal is to prove the following theorem:

Theorem 22.1 *There exists an $O(\log^6 n/\log\log n)$-competitive algorithm for the metrical task system (MTS) problem against an oblivious adversary.*

Recall from the previous lecture the result of [Bartal] that if there is an r-competitive algorithm for n-leaf k-HST's, then a general metric space can be probabilistically approximated with such k-HST's, yielding a randomized algorithm with competitive ratio $O(rk\log n\log_k\min\{\Delta, n\})$ (where Δ is the diameter of the metric space). Thus to show Theorem 22.1, we need an algorithm for $\Omega(\log^2 n)$-HST's with competitive ratio $O(\log^2 n)$.

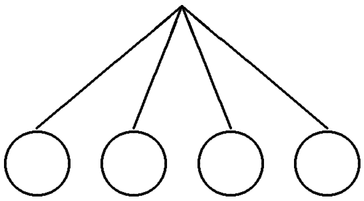

Figure 22.1: A k-HST viewed at the top level

Given a k-HST, we can look at just the immediate children of the root, as in Figure 22.1, treating all descendents of each immediate child as a generic metric space. If the diameter of the whole tree is Δ then the diameter of each smaller space is no greater than Δ/k. Furthermore, the distances between any two points in different spaces are all approximately equal, and we call this distance d. Now, assume we already have an algorithm for each of the smaller spaces, with competitive ratios $r_1 \geq r_2 \geq \cdots r_b$.

The problem now looks almost like the MTS problem on a uniform metric space, except that for a task with cost σ_i in state i, the adversary pays σ_i as usual in state i while an on-line algorithm must pay $r_i\sigma_i$ in state i. This problem is termed Scenario 0. Using the algorithm for MTS on a uniform metric space, it is straightforward to show a competitive ratio of $O(r\log b)$ if the r_i are equal. For the general problem, the k-HST needs to be $L = \log_k\Delta$ levels deep leading to an overall competitive ratio of $(\log n)^L$. It is also possible to show a competitive ratio of $r_1 + O(\log b)$ which leads to an overall competitive ratio of $L\log n$. Unfortunately, Δ appears in the competitive ratio but the competitive ratio in Theorem 22.1 is independent of Δ.

To improve the competitive ratio, we use a work function algorithm. The algorithm is defined by a probability distribution over the metric spaces $(p_1, \ldots, p_b)$. The probabilities are a function of the work function values $(w_1, \ldots, w_b)$ where w_i is the optimal cost of serving the sequence and ending in state i.

Definition 22.2 *An algorithm A has competitive ratio r with potential function Φ against a convex combi-*

nation $\hat{w}$ of work functions if for every task t,

$$c_A(t) + \Delta\Phi \le r\Delta\hat{w} \tag{22.1}$$

It was shown in previous analysis of the metrical task system problem that it suffices to consider requests that have zero cost for all states other that some state i which has cost σ. Such requests are written as (i, σ).

Upon a request (i, σ), p_i, the probability that the algorithm is in state i, must decrease by increasing p_j for some node $j \neq i$. Let p_i be the probability before the request and p'_i be the updated probability after the request. Then:

$$c_A(t) + \Delta\Phi = p'_i \sigma r_i + (p'_i - p_i)(d + \Phi_j - \Phi_i) \tag{22.2}$$

Since the potential function Φ is bounded, we "charge" the change in potential to the movement and call this new problem Scenario 1 which is defined as follows. As in Scenario 0, we consider the MTS problem on a uniform metric space in which the on-line algorithm is charged $r_i\sigma$ for serving the request (i, σ) in state i. In addition, we define a distance ratio s; while an adversary is charged a cost of d for moving between two regions, the on-line algorithm is charged sd when it moves between regions.

To analyze Scenario 1, we make two assumptions. First, if some request (i, σ) causes $p'_i = 0$ (i.e., the algorithm always moves out of state i upon such a request), then we assume that σ is the minimal σ that will force the algorithm to move. Second, we assume that for any state i, if there is a state j such that $w_i = w_j + d_{ij}$, then $p_i = 0$. Any reasonable algorithm must have this property or else it will have an unbounded competitive ratio.

Lemma 22.3 *When a reasonable work-function algorithm is given a task (i, σ), w_i increases by σ.*

This lemma follows directly from the previous two assumptions.

We now define an algorithm for Scenario 1 which we denote Strategy 1. For some odd integer t, the probability distribution for Strategy 1 is:

$$p_j = \frac{1}{b} + \frac{1}{b}\sum_{i=1}^{b}\left(\frac{w_i - w_j}{d}\right)^t \tag{22.3}$$

It is straightforward to show that this algorithm is legal (i.e., $0 \le p_j \le 1$ for all j and that $\sum p_j = 1$) and that it is reasonable as described above.

Theorem 22.4 *Strategy 1 is $(r_1 + 2sb^{1/t}t)$-competitive*

Note that by choosing $t = \log_2 b$, this Strategy 1 yields a competitive ratio of $4s\log_2 b$.

Proof: Let $\Phi = \Phi_l + \Phi_m$ where Φ_l is used to amortize local costs (costs within a region) and Φ_m is used to amortize costs incurred when moving between regions. The potential functions are defined as:

$$\Phi_l = \frac{r_1 d}{2(t+1)b}\sum_{i,j}\left(\frac{w_i - w_j}{d}\right)^{t+1} \tag{22.4}$$

$$\Phi_m = \frac{sd}{2b}\sum_{i,j}\left|\frac{w_i - w_j}{d}\right|^t \tag{22.5}$$

Also, note that

$$\frac{\partial\Phi_l}{\partial w_j} = -\frac{1}{b}\sum_i\left(\frac{w_i - w_j}{d}\right)^t r_1 = -(p_j - \frac{1}{b})r_1 \tag{22.6}$$

For a request $t = (k, \sigma)$, we need to bound the cost of Strategy 1 as follows:

$$c_A(t) + \Delta\Phi \leq r\Delta\hat{w} = r\frac{\sigma}{b} \tag{22.7}$$

Noting that $c_A(t) = p'_k r\sigma + (p_k - p'_k)sd$ and letting y be the value of w_k before the request, we can write

$$c_A(t) + \Delta\Phi = \int_y^{y+\sigma} (p_k r_k + \frac{\partial p_k}{\partial w_k} sd + \frac{\partial \Phi_l}{\partial w_k} + \frac{\partial \Phi_m}{\partial w_k}) dw_k \tag{22.8}$$

Grouping these terms and simplifying, we get:

$$\begin{aligned} \frac{\partial p_k}{\partial w_k} sd + \frac{\partial \Phi_m}{\partial w_k} &= \frac{st}{b}\sum_{i\neq k}\left(\frac{w_i - w_k}{d}\right)^{t-1} + \frac{st}{b}\sum_{w_i<w_k}\left(\frac{w_k - w_i}{d}\right)^{t-1} + \frac{st}{b}\sum_{w_i>w_k}\left(\frac{w_i - w_k}{d}\right)^{t-1} && (22.9)\\ &= 2\frac{st}{b}\sum_{w_i<w_k}\left(\frac{w_k - w_i}{d}\right)^{t-1} && (22.10)\end{aligned}$$

If we let w_a be the maximum w_i then the summation can be bounded:

$$\sum_{w_i<w_k}\left(\frac{w_k - w_i}{d}\right)^{t-1} \leq \sum_{i\neq a}\left(\frac{w_a - w_i}{d}\right)^{t-1} \tag{22.11}$$

If we let $x_i = (w_a - w_i)/d$, we know from the definition of the probability distribution that $\sum x_i^t \leq 1$ and thus to bound $\sum x_i^{t-1}$, the worst case is where the x_i assume the value $b^{-1/t}$ uniformly, thus $\sum x_i^{t-1} \leq b^{1/t}$. Combining (22.6) and (22.10) with (22.8), we get:

$$\begin{aligned} c_A(t) + \Delta\Phi &= \int_y^{y+\sigma} (p_k r_k + \frac{\partial p_k}{\partial w_k} sd + \frac{\partial \Phi_l}{\partial w_k} + \frac{\partial \Phi_m}{\partial w_k}) dw_k && (22.12)\\ &\leq \frac{r_1 + 2stb^{1/t}}{b}\sigma && (22.13)\\ &= (r_1 + 2stb^{1/t})\Delta\hat{w} && (22.14)\end{aligned}$$

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Lemma 22.5 *The potential function used to analyze Strategy 1 is bounded by*

$$\Phi \leq (\frac{r_1}{t+1} + s)d \tag{22.15}$$

This lemma was presented without a proof. It will be needed when analyzing Scenario 0.

References

[Bartal96] Y. Bartal. Probabilistic Approximation of Metric Spaces and its Algorithmic Applications. In *Proc. of the 37th Ann. IEEE Symp. on Foundations of Computer Science*, October 1996.

[BBBT97] Y. Bartal, A. Blum, C. Burch, and A. Tomkins. A polylog(n)-Competitive Algorithm for Metrical Task Systems . In *Proc. of the 29th Ann. ACM Symp. on Theory of Computing*, May 1997.