# Data Structures – LECTURE 15

## Shortest paths algorithms

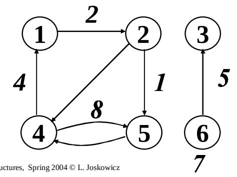
- Properties of shortest paths
- Bellman-Ford algorithm
- Dijsktra's algorithm

Chapter 24 in the textbook (pp 580–599).



## Weighted graphs -- reminder

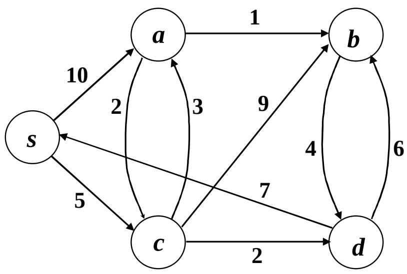
- A *weighted graph* is graph in which edges have *weights* (*costs*) $w(v_i, v_j) > 0$.
- A graph is a weighted graph in which all costs are 1. Two vertices with no edge (path) between them can be thought of having an edge (path) with weight ∞.




*The cost of a path is the sum of the costs of its edges:*



## Example: weighted graph






## Two basic properties of shortest paths

Triangle inequality

Let $G=(V,E)$ be a weighted directed graph, $w: E \rightarrow R$ a weight function and $s \in V$ be a source vertex. Then, for all edges $e=(u,v) \in E$:

$$\delta(s,v) \leq \delta(s,u) + w(u,v)$$

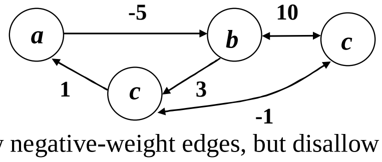
Optimal substructure of a shortest path

Let $p = <v_1, .. v_k>$ be the shortest path between $v_1$ and $v_k$. The sub-path between $v_i$ and $v_j$, where $1 \leq i,j \leq k$, $p_{ij} = <v_i, .. v_j>$ is a shortest path.



## Negative-weight edges

- Shortest paths are well-defined as long as there are no negative-weight cycles. In such cycles, the longer the path, the lower the value → the shortest path has an infinite number of edges!



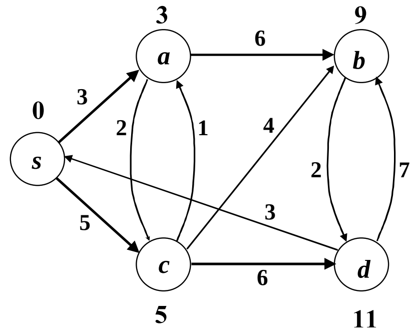

- Allow negative-weight edges, but disallow (or detect) negative-weight cycles!



## Shortest paths and cycles

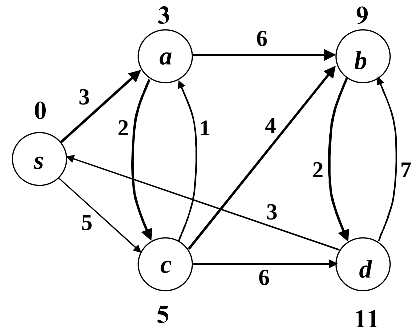
- The shortest path between any two vertices has no positive-weight cycles.
- The representation for shortest paths between a vertex and all other vertices is the same as the one used in the unweigthed BFS: *breath-first tree:* $G_\pi = (V_\pi, E_\pi)$ such that $V_\pi = \{v \in V: \pi[v] \neq \textbf{null}\} \cup \{s\}$ and $E_\pi = \{(\pi[v], v),\ v \in V - \{s\}\}$
- We will prove that a breath-first tree is a shortest-path tree for its root $s$ in which vertices reachable from $s$ are in it and the unique simple path from $s$ to $v$ is shortest.

## Example: shortest-path tree (1)






## Example: shortest-path tree (2)






## Estimated distance from source

- As for BFS on unweighted graphs, we keep a label which is the current best estimate of the shortest distance between $s$ and $v$.
- Initially, $dist[s] = 0$ and $dist[v] = \infty$ for all $v \neq s$, and $\pi[v]$ = ***null.***
- At all times during the algorithm, $dist[v] \geq \delta(s,v)$.
- At the end, $dist[v] = \delta(s,v)$ and $(\pi[v],v) \in E_\pi$



## Edge relaxation

- The process of *relaxing an edge* $(u,v)$ consists of testing whether it can improve the shortest path from $s$ to $v$ so far by going through $u$.

Relax($u$,$v$)
  **if** $dist[v] > dist[u] + w(u,v)$
    **then** $dist[v] \leftarrow dist[u] + w(u,v)$
      $\pi[v] \leftarrow u$



## Properties of shortest paths and relaxation

Triangle inequality

$\forall e = (u,v) \in E$: $\delta(s,v) \leq \delta(s,u) + w(u,v)$

Upper-boundary property

$\forall v \in V$: $dist[v] \geq \delta(s,v)$ at all times, and it keeps decreasing.

No-path property

if there is no path from $s$ to $v$, then $dist[v] = \delta(s,v) = \infty$



## Properties of shortest paths and relaxation

Convergence property

if $s \rightarrow u \rightarrow v$ is a shortest path in $G$ for some $u$ and $v$, and $dist[u] = \delta(s,u)$ at any time prior to relaxing edge $(u,v)$, then $dist[v] = \delta(s,v)$ at all times afterwards.

Path-relaxation property

Let $p = <v_0, .. v_k>$ be the shortest path between $v_0$ and $v_k$. When the edges are relaxed in the order $(v_0, v_1)$, $(v_1, v_2)$, … $(v_{k-1}, v_k)$, then $dist[v_k] = \delta(s,v_k)$.

Predecessor sub-graph property

once $dist[v] = \delta(s,v)$ $\forall v \in V$, the predecessor subgraph is a shortest-paths tree rooted at $s$.

# Two shortest-path algorithms

1. Bellmann-Ford algorithm
2. Dijkstra's algorithm – Generalization of BFS



# Bellman-Ford's algorithm: overview

- Allows negative weights. If there is a negative cycle, returns "a negative cycle exists".
- The idea:
  - There is a shortest path from $s$ to any other vertex that does not contain a non-negative cycle (can be eliminated to produce a shorter path).
  - The maximal number of edges in such a path with no cycles is $|V|–1$, because it can have at most $|V|$ nodes on the path if there is no cycle.
  - $\Rightarrow$ it is enough to check paths of up to $|V|–1$ edges.

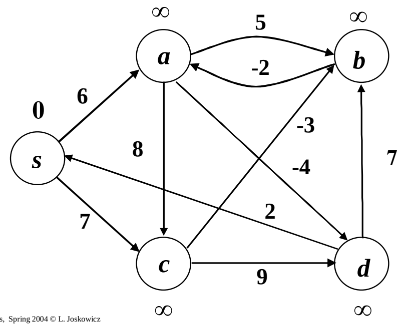


# Bellman-Ford's algorithm

Bellman-Ford$(G, s)$
Initialize$(G, s)$
**for** $i \leftarrow 1$ to $|V| - 1$
  **for** each edge $(u, v) \in E$
    **do if** $dist[v] > dist[u] + w(u, v)$
      $dist[v] \leftarrow dist[u] + w(u, v)$
      $\pi[v] \leftarrow u$
**for** each edge $(u, v) \in E$
  if $dist[v] > d[u] + w(u, v)$ return "negative cycle"



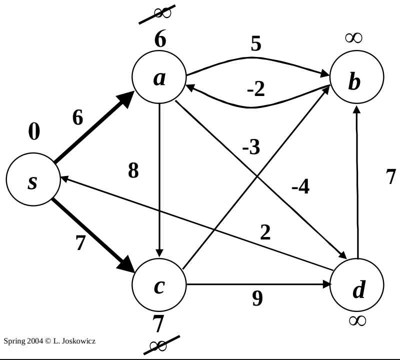
# Example: Bellman-Ford's algorithm (1)




Edge order
$(a,b)$
$(a,c)$
$(a,d)$
$(b,a)$
$(c,b)$
$(c,d)$
$(d,s)$
$(d,b)$
$(s,a)$
$(s,b)$



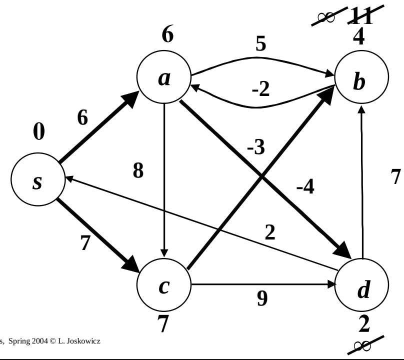
# Example: Bellman-Ford's algorithm (2)




Edge order
$(a,b)$
$(a,c)$
$(a,d)$
$(b,a)$
$(c,b)$
$(c,d)$
$(d,s)$
$(d,b)$
$(s,a)$
$(s,c)$

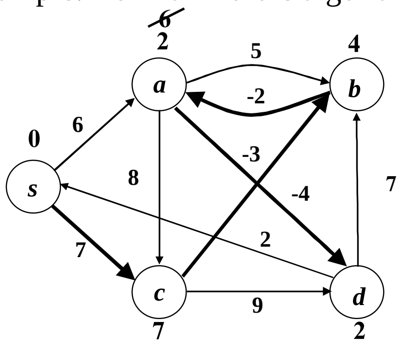


# Example: Bellman-Ford's algorithm (3)




Edge order
$(a,b)$
$(a,c)$
$(a,d)$
$(b,a)$
$(c,b)$
$(c,d)$
$(d,s)$
$(d,b)$
$(s,a)$
$(s,c)$

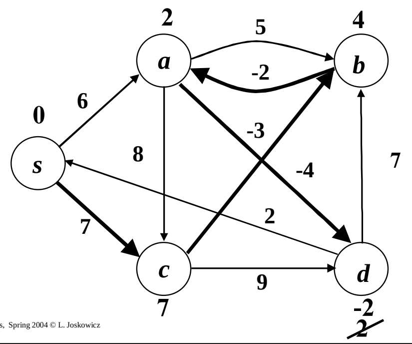
## Example: Bellman-Ford's algorithm (4)




Edge order
($a$,$b$)
($a$,$c$)
($a$,$d$)
($b$,$a$)
($c$,$b$)
($c$,$d$)
($d$,$s$)
($d$,$b$)
($s$,$a$)
($s$,$b$)



## Example: Bellman-Ford's algorithm (5)




Edge order
($a$,$b$)
($a$,$c$)
($a$,$d$)
($b$,$a$)
($c$,$b$)
($c$,$d$)
($d$,$s$)
($d$,$b$)
($s$,$a$)
($s$,$b$)



## Bellman-Ford's algorithm: properties

- The first pass over the edges – only neighbors of $s$ are affected (1-edge paths). All shortest paths with one edge are found.
- The second pass – shortest paths with edges are found.
- After $|V|$-1 passes, all possible paths are checked.
- Claim: we need to update any vertex in the last pass if and only if there is a negative cycle reachable from $s$ in $G$.



## Bellman Ford algorithm: proof (1)

- One direction we already know: if we need to update an edge in the last iteration then there is a negative cycle, because we proved before that if there are no negative cycles, and the shortest paths are well defined, we find them in the $|V|$–1 iteration.
- We claim that if there is a negative cycle, we will discover a problem in the last iteration. Because, suppose there is a negative cycle
- But the algorithm does not find any problem in the last iteration, which means that for all edges, we have that

for all edges in the cycle.



## Bellman Ford algorithm: proof (2)

- Proof by contradiction: for all edges in the cycle
- After summing up over all edges in the cycle, we discover that the term on the left is equal to the first term on the right (just a different order of summation). We can subtract them, and we get that the cycle is actually positive, which is a contradiction.



## Bellman-Ford's algorithm: complexity

- Visits $|V|$–1 vertices → $O(|V|)$
- Performs vertex relaxation on all edges → $O(|E|)$
- Overall, $O(|V|.|E|)$ time and $O(|V|+|E|)$ space.

# Bellman-Ford on DAGs

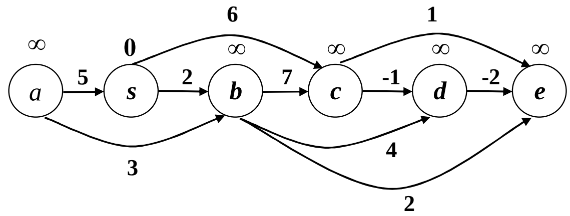
For Directed Acyclic Graphs (DAG), $O(|V|+|E|)$ relaxations are sufficient when the vertices are visited in topologically sorted order:

DAG-Shortest-Path($G$)

1. Topologically sort the vertices in $G$
2. Initialize $G$ ($dist[v]$ and $\pi(v)$) with $s$ as source.
3. **for** each vertex $u$ in topologically sorted order **do**
4. **for** each vertex $v$ incident to $u$ **do**
5. Relax($u$,$v$)



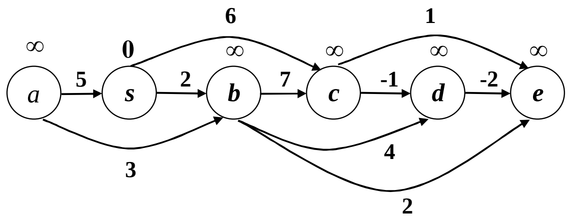
# Example: Bellman-Ford on a DAG (1)




Vertices sorted from left to right

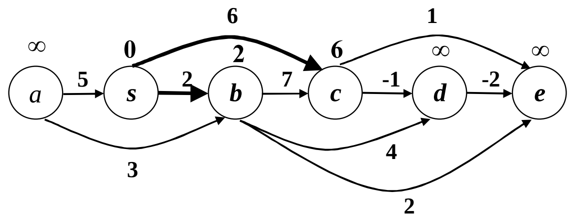


# Example: Bellman-Ford on a DAG (2)



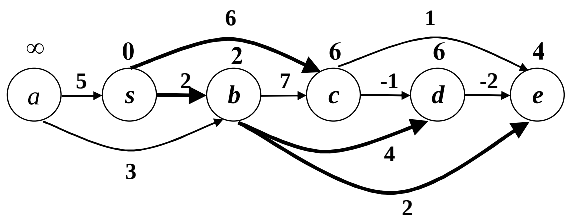



# Example: Bellman-Ford on a DAG (3)



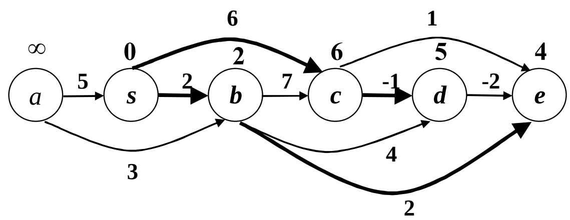



# Example: Bellman-Ford on a DAG (4)

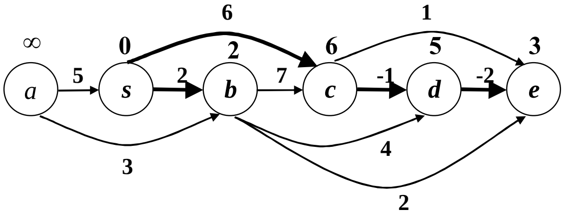





# Example: Bellman-Ford on a DAG (5)

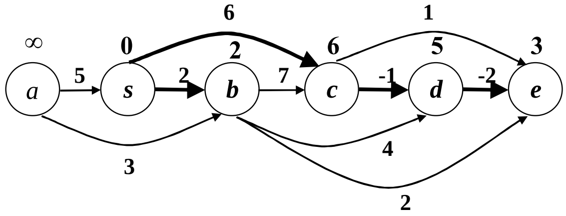


## Example: Bellman-Ford on a DAG (6)






## Example: Bellman-Ford on a DAG (7)






## Bellman-Ford on DAGs: correctness

<u>Path-relaxation property</u>

Let $p = <v_0, .. v_k>$ be the shortest path between $v_0$ and $v_k$. When the edges are relaxed in the order $(v_0, v_1), (v_1, v_2), \ldots (v_{k-1}, v_k)$, then $dist[v_k] = \delta(s,v_k)$.

In a DAG, we have the correct ordering!

Therefore, the complexity is $O(|V|+|E|)$.



## Dijkstra's algorithm: overview

<u>Idea</u>: Do the same as BFS for unweighted graphs, with two differences:

- use the cost as the distance function
- use a minimum priority queue instead of a simple queue.

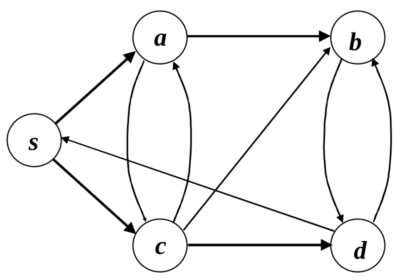


## The BFS algorithm

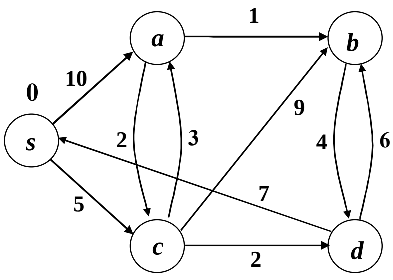
<u>BFS(*G*, *s*)</u>
*label*[*s*] ← *current; dist*[*s*] = 0*;* π[*s*] = ***null***
**for** all vertices *u* in *V* – {*s*} **do**
  *label*[*u*] ← *not_visited; dist*[*u*] = ∞*;* π[*u*] = ***null***
EnQueue(*Q*,*s*)
**while** *Q* is not empty **do**
  *u* ← DeQueue(*Q*)
  **for** each *v* that is a neighbor of *u* **do**
    **if** *label*[*v*] = *not_visited* **then** *label*[*v*] ← *current*
    *dist*[*v*] ← *dist*[*u*] + 1; π[*v*] ← *u*
    EnQueue(*Q*,*v*)
  *label*[*u*] ← *visited*



## Example: BFS algorithm



## Example: Dijkstra’s algorithm



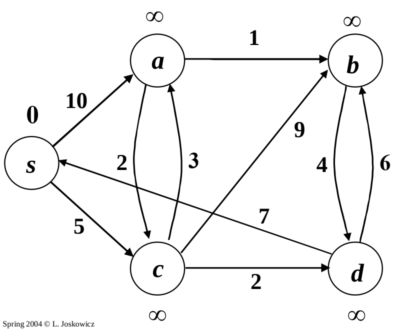



## Dijkstra’s algorithm

Dijkstra(*G*, *s*)
*label*[*s*] ← *current; dist*[*s*] = 0; π[*u*] = ***null***
**for** all vertices *u* in *V* – {*s*} **do**
  *label*[*u*] ← *not_visited; dist*[*u*] = ∞; π[*u*] = ***null***
*Q* ← *s*
**while** *Q* is not empty **do**
  *u* ← Extract-Min(*Q*)
  **for** each *v* that is a neighbor of *u* **do**
    ~~**if** *label*[*v*] = *not_visited* **then** *label*[*v*] ← *current*~~
    **if** *d*[*v*] > *d*[*u*] + *w*(*u*,*v*)
      **then** *d*[*v*] ← *d*[*u*] + *w*(*u*,*v*); π[*v*] = *u*
    Insert-Queue(*Q*,*v*)
  ~~*label*[*u*] = *visited*~~

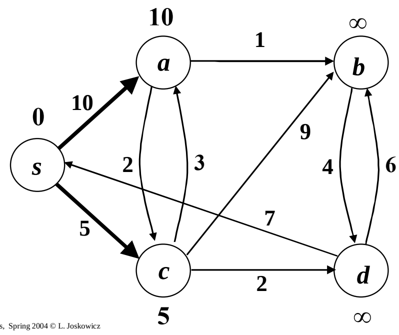


## Example: Dijkstra’s algorithm (1)



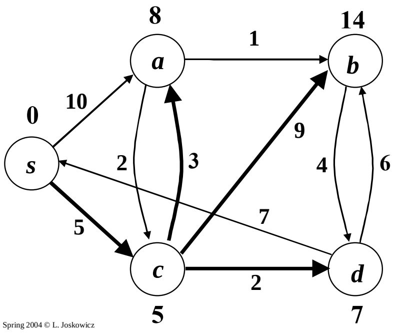



## Example: Dijkstra’s algorithm (2)



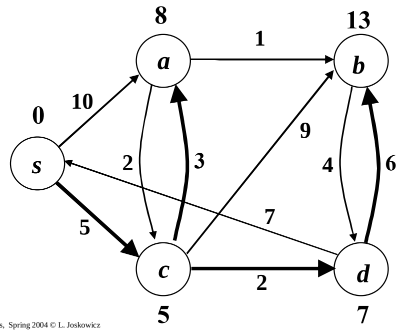



## Example: Dijkstra’s algorithm (3)

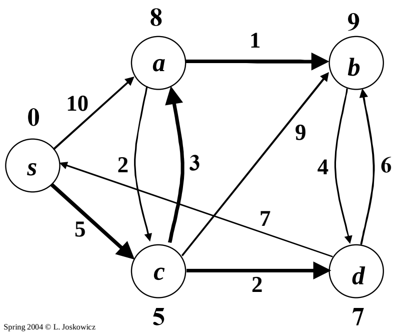





## Example: Dijkstra’s algorithm (4)

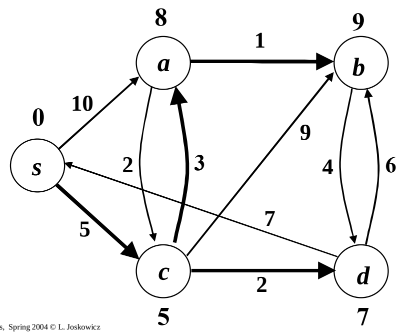


## Example: Dijkstra's algorithm (5)






## Example: Dijkstra's algorithm (6)






## Dijkstra's algorithm: correctness (1)

Theorem: Upon termination of the Dijkstra's algorithm, for each $dist[v] = \delta(s,v)$ for each vertex $v \in V$,

Definition: a path from *s* to *v* is said to be a *special* path if it is the shortest path from *s* to *v* in which all vertices (except maybe for *v*) are inside *S*.

Lemma: At the end of each iteration of the **while** loop, the following two properties hold:

1. For each $w \in S$, *dist*[*w*] is the length of the shortest path from *s* to *w* which stays inside *S*.
2. For each $w \in V–S$ , *dist*(*w*) is the length of the shortest **special** path from *s* to *w*.

The theorem follows when *S* = *V*.



## Dijkstra's algorithm: correctness (2)

Proof: by induction on the size of *S*.

- For |*S*|=1, it is clearly true: *dist*[*v*] = ∞ except for the neighbors of *s*, which contain the length of the shortest special path.
- Induction step: suppose that in the last iteration node *v* was added added to *S*. By the induction assumption, *dist*[*v*] is the length of the shortest special path to *v*. It is also the length of the general shortest path to *v*, since if there is a shorter path to *v* passing through nodes of *S*, and *x* is the first node of *S* in that path, then *x* would have been selected and not *v*. So the first property still holds.



## Dijkstra's algorithm: correctness (3)

Property 2: Let $x \in S$. Consider the shortest new special path to *w*

If it doesn't include *v*, *dist*[*x*] is the length of that path by the induction assumption from the last iteration since *dist*[*x*] did not change in the final iteration.

If it does include *v*, then *v* can either be a node in the middle or the last node before *x*. Note that *v* cannot be a node in the middle since then the path would pass from *s* to *v* to *y* in *S*, but by property 1, the shortest path to *y* would have been inside *S* → *v* need not be included.

If *v* is the last node before *x* on the path, then *dist*[*x*] contains the distance of that path, by the substitution *dist*[*x*] = *dist*[*v*] + *w*(*v*,*x*) in the algorithm.



## Dijkstra's algorithm: complexity

- The algorithm performs |*V*| Extract-Min operations and |*E*| Insert-Queue operations.
- When the priority queue is implemented as a heap, insert takes $O(\lg|V|)$ and Extract-Min takes $O(\lg(|V|)$. The total time is $O(|V|\lg|V|) + O(|E|\lg|V|) = O(|E|\lg|V|)$
- When $|E| = O(|V|^2)$, this is not optimal. In this case, there are many more insert than extract operations.
- Solution: Implement the priority queue as an array! Insert will take $O(1)$ and Extract-Min $O(|V|)$ →

$$O(|V|^2) + O(|E|) = O(|V|^2)$$

which is better than the heap as long as |*E*| is $O(|V|^2/\lg(|V|))$.

## Application: difference constraints

- Given a system of $m$ difference constraints over $n$ variables, find a solution if one exists.

$$x_i - x_j \le b_k$$

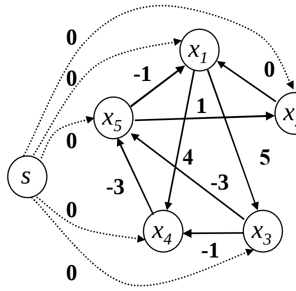
for $1 \le i, j \le n$ and $1 \le k \le m$

- Constraint graph $G$: each variable $x_i$ is a vertex, each constraint $x_i - x_j \le b_k$ is a directed edge from $x_i$ to $x_j$ with weight $b_k$.
- When $G$ does not have negative cycles, the minimum path distances of the vertices are the solution to the system of constraint differences.



## Example: difference constraints (1)

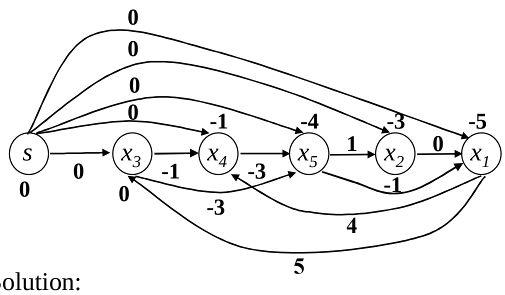
$x_1 - x_2 \le 0$
$x_1 - x_5 \le -1$
$x_2 - x_5 \le 1$
$x_3 - x_1 \le 5$
$x_4 - x_1 \le 4$
$x_4 - x_3 \le -1$
$x_5 - x_3 \le -3$
$x_5 - x_4 \le -3$




Solution:
$\boldsymbol{x} = (-5,-3,0,-1,-4)$



## Example: difference constraints (2)




Solution:
$\boldsymbol{x} = (-5,-3,0,-1,-4)$



## Why does this work?

Theorem: Let $Ax \le b$ be a set of $m$ difference constraints over $n$ variables, and $G=(V,E)$ its corresponding constraint graph. If $G$ has no negative weight cycles, then

$$\boldsymbol{x} = (\delta(v_0,v_1),\delta(v_0,v_2), \ldots ,\delta(v_0,v_n))$$

is a feasible solution for the system. If $G$ has a negative cycle, then there is no feasible solution.

Proof outline: For all edges $(v_i,v_j)$ in $E$:

$$\delta(v_0,v_j) \le \delta(v_0,v_i) + w(v_i,v_j)$$
$$\delta(v_0,v_j) - \delta(v_0,v_i) \le w(v_i,v_j)$$
$$x_j \quad - \quad x_j \quad \le w(v_i,v_j)$$



## Summary

- Solving the shortest-path problem on weighted graphs is performed by relaxation, based on the path triangle inequality: for all edges $e=(u,v)\in E$:

$$\delta(s,v) \le \delta(s,u) + w(u,v)$$

- Two algorithms for solving the problem:
  - Bellman Ford: for each vertex, relaxation on all edges. Takes $O(|E|.|V|)$ time. Works on graphs with non-negative cycles.
  - Dijkstra: BFS-like, takes $O(|E|\lg|V|)$ time.