Possible Winners When New Alternatives Join: New Results Coming Up!

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ABSTRACT

In a voting system, sometimes multiple new alternatives will join the election after the voters' preferences over the initial alternatives have been revealed. Computing whether a given alternative can be a co-winner when multiple new alternatives join the election is called the *possible co-winner with new alternatives (PcWNA)* problem and was introduced by Chevaleyre et al. [6]. In this paper, we show that the PcWNA problems are NP-complete for the Bucklin, Copeland$_0$, and maximin (a.k.a. Simpson) rule, even when the number of new alternatives is no more than a constant. We also show that the PcWNA problem can be solved in polynomial time for plurality with runoff. For the approval rule, we examine three different ways to extend a linear order with new alternatives, and characterize the computational complexity of the PcWNA problem for each of them.

Categories and Subject Descriptors

J.4 [**Computer Applications**]: Social and Behavioral Sciences–Economics; I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems

General Terms

Algorithms, Economics, Theory

Keywords

Computational social choice, possible co-winner with new alternatives

1. INTRODUCTION

In many real-life situations, multiple voters have to choose a common alternative out of a set that can grow during the process. For instance, when a committee wants to decide which proposal should be approved, some applications might arrive late (due to unexpected delay in the mailing system, etc). Suppose that we have already elicited the preference of the voters (members of the committee) on the initial proposals. It is important for the applicants to know whether they are already out (so that they can submit the same proposal to other founding sources right away without waiting for the committee members to make the final decision). A recent paper by Chevaleyre et al. [6] considers the following problem: *suppose that the voters' preferences about a set of initial alternatives have already been elicited, and we know that a given number k of new alternatives will join the election; we ask who among the initial alternatives can possibly win the election in the end*. This problem is a special case of the *possible winner problem* [18, 21, 20, 3, 4, 2], restricted to the case where the incomplete profile consists of a collection of full rankings over the initial alternatives (nothing being known about the voters' preferences about the new alternatives). It is somehow dual of another special case of the problem where the incomplete profile consists of a collection of full rankings over all alternatives for a subset of voters (nothing being known about the remaining voters' preferences), which itself is equivalent to the coalitional manipulation problem. The problem is also related to control by adding candidates [1, 11, 14, 12], as discussed in [6].

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Ideally, given a voting rule, we would hope to find a polynomially-computable characterization which would allow us to quickly identify the possible (co)winners given an incomplete profile P and a number of new alternatives. Chevaleyre *et al.* [6] give such characterizations for plurality and Borda for an arbitrary number of new candidates, as well as for K-approval when there is a single new candidate. They show that these positive results do not extend to scoring rules in general, not even to K-approval, and show that computing possible (co)winners for 3-approval is NP-hard with three new candidates, as well as for some more sophisticated scoring rules, for a single new candidate. These results were further extended in [7], where a polynomial algorithm (but not an easy characterization) was proposed for 2-approval, as well as for K-approval for 2 new candidates.

The results given in [6] and [7] do not go beyond scoring rules. In this paper we go further by considering major voting rules that are outside the family of scoring rules, namely *approval*, *Bucklin*, *Copeland*, *maximin* and *plurality with runoff*. We will give two positive results, namely polynomially-computable characterization of possible (co)winners with new alternatives, for plurality with runoff and, with some specific assumptions we shall discuss later, for approval. However, for all other rules considered in this paper, we will show that finding such a characterization is hopeless, as we show that the possible (co)winner problem with new alternatives for these rules is NP-hard.

The reason why it is worth exploring the computational complexity of the possible (co)winner problem with new alternatives for various voting rules is threefold. First, it helps understanding the various possible (co)winner problems better, by comparing our results to complexity results of the possible (co)winner in the general case [21] as well as as in the specific case corresponding to the unweighted coalitional manipulation problem (see *e.g.*, [13, 15]). Second, these results help deciding which voting rules to apply in situations where we know beforehand that new candidates may come after the initial ones and where we want to know which of the initial ones can win the election. Voting rules for which we have an easy (polynomial) way to compute these possible winners are better suited to this class of situations. Third, on the other hand, hardness results can also be considered positive in settings where we want a voting rule to be hard to control by adding candidates *without the chair knowing a priori the voters' preferences on these candidates*. (We shall say more about this in Section 3).

We start by giving some background in Section 2. In Section 3 we recall the possible co-winner problem with respect to the addition of new alternatives (PcWNA). Each of the following sections is devoted to the PcWNA problem for a specific voting rule.

In Section 4 we focus on approval voting. Since the notion of a complete profile (including the new alternatives) extending a partial profile over the initial alternatives is not straightforward, we investigate three possible definitions, which we think are the three most reasonable definitions. To the best of our knowledge, two of these definitions are new. We show that PcWNA problems are trivial for two of these definitions, and NP-complete for the third one.

In Sections 5, 6 and 7 we show that the problem is NP-complete for, respectively, the Bucklin rule, the Copeland rule, and the maximin (a.k.a. Simpson) rule, and finally in Section 8 we focus on plurality with runoff, for which we give a polynomially computable characterization.

2. PRELIMINARIES

Let $\mathcal{C}$ be the set of *alternatives* (or candidates), with $|\mathcal{C}| = m$. Let $\mathcal{I}(\mathcal{C})$ denote the set of votes. Most often, the set of votes is the set of all linear orders over $\mathcal{C}$. An *n-profile* P is a collection of n votes for some $n \in \mathbb{N}$, that is, $P \in \mathcal{I}(\mathcal{C})^n$. A *voting rule* r is a mapping that assigns to each profile a set of winning alternatives[1], that is, r is a mapping from $\{\emptyset\} \cup \mathcal{I}(\mathcal{C}) \cup \mathcal{I}(\mathcal{C})^2 \cup \ldots$ to $2^{\mathcal{C}}$. For any profile P, the alternatives in $r(P)$ are called *co-winners* for P. If $r(P) = \{c\}$, then c is the *unique winner* for P.

Some common voting rules are listed below. For all of them (except the approval rule), $\mathcal{I}(\mathcal{C})$ is the set of all linear orders over $\mathcal{C}$; for the approval rule, the set of votes is the set of all subsets of $\mathcal{C}$, that is, $\mathcal{I}(\mathcal{C}) = \{S : S \subseteq \mathcal{C}\}$.

• *(Positional) scoring rules*: Given a *scoring vector* $\vec{v} = (v(1), \ldots, v(m))$, for any vote $V \in L(\mathcal{C})$ and any $c \in \mathcal{C}$, let $s(V, c) = v(j)$, where j is the rank of c in V. For any profile $P = (V_1, \ldots, V_n)$, let $s(P, c) = \sum_{i=1}^{n} s(V_i, c)$. The rule will select $c \in \mathcal{C}$ so that $s(P, c)$ is maximized. Some examples of positional scoring rules are *Borda*, for which the scoring vector is $(m-1, m-2, \ldots, 0)$; *l-approval* ($l \leq m$), for which the scoring vector is $v(1) = \ldots = v(l) = 1$ and $v_{l+1} = \ldots = v_m = 0$; and *plurality*, for which the scoring vector is $(1, 0, \ldots, 0)$.

• *Approval*: Each voter submits a set of alternatives (that is, the alternatives that are "approved" by the voter). The winner is the alternative approved by the largest number of voters. Note that the approval rule is different from the l-approval rule, in that for the l-approval rule, a voter must approve l alternatives, whereas for the approval rule, a voter can approve an arbitrary number of alternatives.

• *Bucklin*: The Bucklin score of an alternative c, denoted by $B_P(c)$ is the smallest number t such that more than half of the votes rank c among top t positions. A Bucklin winner has the lowest Bucklin score and is ranked within top $B_P(c)$ for most times.

[1]Such a function is often called a *voting correspondence* rather than a voting rule. We will however stick to the terminology "rule" throughout the paper.

• *Copeland$_\alpha$* ($0 \leq \alpha \leq 1$): For any two alternatives c_i and c_j, we can simulate a *pairwise election* between them, by seeing how many votes prefer c_i to c_j, and how many prefer c_j to c_i; the winner of the pairwise election is the one preferred more often. Then, an alternative receives one point for each win in a pairwise election, α points for each tie, and zero point for each loss. The alternatives that have the highest score win.

• *maximin (a.k.a. Simpson)*: Let $N_P(c_i, c_j)$ denote the number of votes that rank c_i ahead of c_j in P. The maximin score of alternative $c \in C$ in profile P is defined as $Sim_P(c) = \min\{N_P(c, c') : c' \in C \setminus \{c\}\}$. A maximin winner maximizes the maximin score.

• *Plurality with runoff*: The election has two rounds. In the first round, all alternatives are eliminated except the two with the highest plurality scores. In the second round (runoff), the winner is the alternative that wins the pairwise election between them. Here we use the *parallel-universe tie-breaking mechanism* [8], where an alternative c is a co-winner, if there exists a way to break ties in both rounds to make c win.

In this paper, all NP-hardness results are proved by reductions from the EXACT COVER BY 3-SETS problem (denoted by X3C) or the 3-DIMENSIONAL MATCHING problem (denoted by 3DM). An instance $I = (\mathcal{S}, \mathcal{V})$ of X3C consists of a set $\mathcal{V} = \{v_1, \ldots, v_{3q}\}$ of $3q$ elements and $t \geq q$ 3-sets $\mathcal{S} = \{S_1, \ldots, S_t\}$ of $\mathcal{V}$, i.e., for any $i \leq t$, $S_i \subseteq \mathcal{V}$ and $|S_i| = 3$. Without loss of generality, we assume that for each $v \in \mathcal{V}$, there exists $S \in \mathcal{S}$ such that $v \in S$. For any $v \in \mathcal{V}$, let $d_I(v)$ denote the number of 3-sets containing element v in instance I. Let $\Delta(I) = \max_{v \in \mathcal{V}} d_I(v)$. We are asked whether there exists a subset $J \subseteq \{1, \ldots, t\}$ such that $|J| = q$ and $\bigcup_{j \in J} S_j = \mathcal{V}$ (indeed, the sets S_j for $j \in J$ form a partition of $\mathcal{V}$). This problem is known to be NP-complete, even if $\Delta(I) \leq 3$ (problem [SP2] page 221 in [16]). In this paper, we will use a special case of 3DM that is also a special case of X3C, defined as follows.[2] Given A, B, X, where $A = \{a_1, \ldots, a_q\}$, $B = \{b_1, \ldots, b_q\}$, $X = \{x_1, \ldots, x_q\}$, $T \subseteq A \times B \times X$, $T = \{S_1, \ldots, S_t\}$ with $t \geq q$. We are asked whether there exists $M \subseteq T$ such that $|M| = q$ and for any $(a_1, b_1, x_1), (a_2, b_2, x_2) \in M$, we have $a_1 \neq a_2$, $b_1 \neq b_2$, and $x_1 \neq x_2$. That is, M corresponds to an exact cover of $\mathcal{V} = A \cup B \cup X$. This problem with the restriction where no element of $A \cup B \cup X$ occurs in more than 3 triples (i.e., $\Delta(I) \leq 3$) is known to be NP-complete (problem [SP1] page 221 in [16]).

To prove our NP-hardness results, we first prove that another useful special case of 3DM (as well as X3C) remains NP-complete.

Proposition 1 3DM *is* NP*-complete, even if q is even, $t = 3q/2$, and $\Delta(I) \leq 6$.*

PROOF. Let $I = (T, A \times B \times X)$ be an instance of 3DM with $A = \{a_1, \ldots, a_q\}$, $B = \{b_1, \ldots, b_q\}$, $X = \{x_1, \ldots, x_q\}$, $T \subseteq A \times B \times X$, $T = \{S_1, \ldots, S_t\}$ and $\Delta(I) \leq 3$. We next show how to build an instance $I' = (T', A' \times B' \times X')$ of 3DM in polynomial time, with $|A'| = |B'| = |X'| = q'$, $T' \subseteq A' \times B' \times X'$ and $|T'| = t'$ such that q' is even, $t' = 3q'/2$, and $\Delta(I') \leq 6$.

• If q is odd, then we add to the instance 3 new elements $\{a'_1, b'_1, x'_1\}$ with $A' = A \cup \{a'_1\}$, $B' = B \cup \{b'_1\}$, $X' = X \cup \{x'_1\}$ and one new triplet (a'_1, b'_1, x'_1).

• Suppose that q is even. If $t > 3q/2$, then we add $6(t - 3q/2)$ new elements $\{a'_1, \ldots, a'_{2(t-3q/2)}\}$ to A, $\{b'_1, \ldots, b'_{2(t-3q/2)}\}$ to B, $\{x'_1, \ldots, x'_{2(t-3q/2)}\}$ to X and $2(t - 3q/2)$ new triples $\{S'_1, \ldots, S'_{2(t-3q/2)}\}$, where for any $i \leq 2(t - 3q/2)$, $S'_i = (a'_i, b'_i, x'_i)$. If $t < 3q/2$, then we add $3q/2 - t$ dummy triples to T by duplicating $3q/2 - t$ triples of T once each. We note that $t \geq q$ implies that $t \geq 3q/2 - t$.

[2]Generally, 3DM is not a special case of X3C.

It is easy to check that in I', q' is even, $t' = 3q'/2$, and $\Delta(I') \leq 6$. The size of the input of the new instance is polynomial in the size of the input of the old instance. Moreover, I is a yes-instance if and only if I' is also a yes-instance. □

3. POSSIBLE (CO)WINNERS WITH NEW ALTERNATIVES

Let $\mathcal{C}$ denote the set of original alternatives, let Y denote the set of new alternatives. For any linear order V over $\mathcal{C}$, a linear order V' over $\mathcal{C} \cup Y$ *extends* V, if in V' the pairwise comparison between any pair of alternatives in $\mathcal{C}$ is the same as in V. That is, for any $c, d \in \mathcal{C}$, $c \succ_V d$ if and only if $c \succ_{V'} d$.

Given a voting rule r, an alternative c, and a profile P over $\mathcal{C}$, we are asked whether there exists a profile P' over $\mathcal{C} \cup Y$ such that P' is an extension of P and $c \in r(P')$. This problem is called the *possible co-winner with new alternatives (PcWNA)* problem [6, 7].

Similarly, we let *PWNA* denote the problem in which we are asked whether c is a possible (unique) winner, that is, $r(P') = \{c\}$. Up to now, the PcWNA and PWNA problems are well-defined for all voting rules studied in this paper (except the approval rule). For the approval rule, we will introduce three types of extension, and discuss the computational complexity of the PcWNA and PWNA problems under these extensions.

We denote by $\text{PWNA}_r(P, k)$ (respectively $\text{PcWNA}_r(P, k)$) the set of possible winners (respectively co-winners) for voting rule r and profile P with respect to the addition of k new alternatives.

It is straightforward to check that the PcWNA (respectively, PWNA) problems for all voting rules studied in this paper are in NP, because given an extension of a profile P, it takes polynomial time to verify if the given alternative c is a co-winner (respectively, the unique winner) for all rules studied in this paper. Therefore, in this paper, we do not show that PcWNA and PWNA are in NP for individual voting rules. (That is, we only show either polynomiality or NP-hardness proofs.)

Chevaleyre *et al.* [6, 7] discuss the relationship between the P(c)WNA problem and two related problems, namely control via adding candidates and candidate cloning. It is argued that the main difference between the three problems is that in the problem of control via adding candidates, the chair knows how the voters would rank the new candidates that can possibly be added by her; in the problem of candidate cloning, the chair only knows that every voter will order all the clones of a candidate contiguously in her vote, that is, every voter's preferences between a clone of c and another candidate d must be the same as her preferences between c and d; whereas in the P(c)WNA problem, the chair does not have any information about how the voters would rank the new candidates.

Even though it has been defined primarily as a problem dealing with voting with incomplete knowledge, the possible co-winner problem with new alternatives *can also be seen as a constructive control problem*, for the class of situations where the chair can add a number of new candidates without knowing how the voters will rank them: if the chair's preferred candidate x is not a co-winner for the current profile P, the chair has an incentive to add a number of new candidates for which x becomes a possible co-winner of the profile before the new alternatives are added. Of course the chair cannot guarantee that x must be a co-winner after the new alternatives are added[3], but at least x has some hope to win. The chair could find, even further, the number of new candidates k such that not only x becomes a possible co-winner, but also such that the number of possible co-winners is as low as possible.

[3]This actually corresponds to the *necessary co-winner problem*, to which the answer is trivial in the setting of this paper.

4. APPROVAL

Since the input of the approval rule is different from the input of other voting rules studied in this paper, we have to define the set of possible extensions of an approval profile over $\mathcal{C}$. Let $P_\mathcal{C} = (V_1, \ldots, V_n)$ be an approval profile over $\mathcal{C}$, where each V_i is a subset of $\mathcal{C}$. An extension of $P_\mathcal{C}$ over $\mathcal{C} \cup Y$ is a collection $(V'_1, \ldots, V'_n)$ where $V'_i \subseteq \mathcal{C} \cup Y$ is an extension of V_i. Now, we define what it means to say that $V' \subseteq \mathcal{C} \cup Y$ is an extension of $V \subseteq \mathcal{C}$. We can think of three natural definitions as follows.

Definition 1 (extension of an approval vote, definition 1) *$V' \subseteq \mathcal{C} \cup Y$ is an extension of $V \subseteq \mathcal{C}$ if $V' \cap \mathcal{C} = V$.*

In other words, under this definition, V' is an extension of V if $V' = V \cup Y'$, where $Y' \subseteq Y$. This definition coincides with the definition used in [19] (Definition 4.3) for the control of approval voting by adding candidates. The problem with Definition 1 is that it assumes that any alternative approved in V is still approved in V'. However, in some contexts, extending the choice with alternatives of Y may change the "approval threshold". Moreover, since we have more alternatives, this threshold should either stay the same or move upward: some alternatives that were approved initially may become disapproved. This leads to the following definition of extension.

Definition 2 (extension of an approval vote, definition 2) *$V' \subseteq \mathcal{C} \cup Y$ is an extension of $V \subseteq X$ if one of the following conditions holds: (1) $V = V'$; (2) $V' \cap Y \neq \emptyset$ and $V' \cap \mathcal{C} \subseteq V$.*

Lastly, we may also allow the acceptance threshold to move downward, even though the set of alternatives grows, especially in the case where the new alternatives are particularly bad, thus rendering some alternatives in $\mathcal{C}$ acceptable after all. This leads to the third definition of extension.

Definition 3 (extension of an approval vote, definition 3) *$V' \subseteq \mathcal{C} \cup Y$ is an extension of $V \subseteq \mathcal{C}$ if one of the following conditions holds: (1) $V' \cap \mathcal{C} \subset V$ and $V' \cap Y \neq \emptyset$; (2) $V \subset V' \cap \mathcal{C}$, and $Y \setminus V' \neq \emptyset$; (3) $V' \cap \mathcal{C} = V$.*

Under Definition 3, either the threshold moves upward, in which case all alternatives which were disapproved in V are still disapproved in V', and obviously, at least one alternative in Y should be approved; or the threshold moves downward, in which case all alternatives that were approved in V are still approved in V', and obviously not all alternatives in Y should be approved. Note that in the case where $V' \cap \mathcal{C} = V$, the threshold can move upward, or downward, or remain the same[4].

Let us give a brief summary of the three definitions of extension. Definition 1 assumes that the threshold cannot move; Definition 2 assumes that the threshold can stay the same or move upward (because the set of alternatives grows); and Definition 3 assumes that the threshold can stay the same, move upward, or move downward. Next, we show an example that illustrates these definitions. Let $\mathcal{C} = \{a, b, c, d\}$, $Y = \{y_1, y_2\}$, and $V = \{a, b\}$.

[4]The rationale behind Definition 3 is that the threshold may depend on the average quality of the alternatives, and therefore may go down after some bad new alternatives have been added. For instance, suppose a voter hates red meat, and has the preference relation `tofu` $\succ$ `fish` $\succ$ `chicken` $\succ$ `beef` $\succ$ `mutton`; if the initial set of alternatives is {`tofu, fish, chicken`}, it is perfectly reasonable that he should approve {`tofu, fish` }, while he would approve {`tofu, fish, chicken`} after `beef` and `mutton` have been added to the set of alternatives. This is perfectly in agreement with the notion of sincere ballot in approval voting (see, *e.g.*, [5, 10, 11] and references therein).

• $V_1' = \{a, b\}$ and $V_2' = \{a, b, y_1\}$ are extensions of V under all three definitions;

• $V' = \{a, y_1\}$ is an extension of V under definitions 2 and 3 but not under definition 1 (the threshold has moved upward, since b was approved in V and is no longer approved in V');

• $V' = \{a, b, c, y_1\}$ is an extension of V under definition 3 but neither under definitions 1 nor 2 (the threshold has moved downward, since c was not approved in V and becomes approved in V'—note that, intuitively, y_2 must be a very unfavorable alternative for this to happen);

• $V' = \{a, b, c\}$ is an extension of V under definitions 3 but neither under definitions 1 nor 2, for the same reason as above;

• $V' = \{a\}$ is not an extension of V under any of the definitions: to have b disapproved in V' and approved in V, the threshold has to move upward, which cannot be the case if no alternative of Y is approved;

• $V' = \{a, b, c, y_1, y_2\}$ is not an extension of V under any of the definitions: to have c disapproved in V and approved in V', the threshold has to move downward, which cannot be the case where all alternatives in Y are disapproved;

• $V' = \{a, c, y_1\}$ is not an extension of V under any of the definitions: the threshold cannot simultaneously move upward and downward.

It is straightforward to check that the PcWNA and PWNA problems are in P for approval under definition 1: an alternative $c \in \mathcal{C}$ is a possible (co-)winner in P if and only if it is a (co-)winner for approval in P (this is because for any $V \in P$, the scores of alternatives in $\mathcal{C}$ will not change from V to its extension V'). However, when we adopt definition 2 of extension, the problems become NP-complete.

Theorem 1 *Under Definition 2, the PcWNA and PWNA problems are* NP*-complete for the approval rule.*

PROOF. We first prove the hardness of the PcWNA problem by a reduction from X3C. For any X3C instance $I = (\mathcal{S}, \mathcal{V})$, we construct the following PcWNA instance.

Alternatives: $\mathcal{V} \cup \{c\} \cup Y$, where $Y = \{y_1, \ldots, y_{t-q}\}$.

Votes: for any $i \leq t$, we have a vote $V_i = S_i$; and we have an additional vote $V_{t+1} = \{c\}$. That is, $P = (V_1, \ldots, V_t, V_{t+1})$.

Suppose the X3C instance has a solution, denoted by $\{S_{i_1}, \ldots, S_{i_q}\}$. Then, take the following extension P' of P: for any $j \leq q$, let $V'_{i_j} = V_{i_j}$. For any $i \leq t$ such that $i \neq i_j$ for all $j \leq q$, we let V'_i be a singleton containing exactly one of the new alternatives. Let $V'_{t+1} = \{c\}$. For any $v \in \mathcal{V}$, because v appears exactly in one S_{i_j}, v is approved by exactly one voter. So is c. Now, there are exactly $t - q$ votes V_i where i is not equal to one of the i_j's. Therefore, the total approval score of the new alternatives is $t - q$, and it suffices to approve every new alternative exactly once. Therefore c is a co-winner in P', and thus a possible co-winner in P.

Conversely, suppose c is a possible co-winner for P and let P' be an extension of P for which c is a co-winner. We note that c is approved at most once in P'. Therefore, every alternative in $\mathcal{V} \cup Y$ must be approved at most once. Without loss of generality, assume that every vote V'_i in P' is either of the form V_i or of the form $\{y_j\}$ (if not, remove every alternative (except one y_j) from V'_i; c will still be a co-winner in the resulting profile). Since we have $t - q$ new alternatives, each being approved at most once in P', we have at least q votes V'_i in P' such that $V'_i = V_i$. If we had more than q votes V'_i such that $V'_i = V_i$, then more than $3q$ points would be distributed to $3q$ alternatives and one of them would get at least 2, which means that c would not be a co-winner in P'. Therefore we have exactly q votes V'_i such that $V'_i = V_i$, and $3q$ points distributed to $3q$ alternatives; since none of them gets more than one point, they get one point each, which implies that the collection of all S_i such that $V_i = V'_i$ forms an exact cover of C.

For the PWNA problem, we add one more vote $V_{t+2} = \{c\}$ to the profile P. □

Now, let us consider Definition 3. Notice that the profile P' where every voter adds c to her vote (if she was not already voting for c) is an extension of P, and obviously c is a co-winner in P'. Therefore, every alternative in $\mathcal{C}$ is a possible co-winner for P, which trivialize the problem.

5. BUCKLIN

Theorem 2 *The PWNA and PcWNA problems are* NP*-complete for Bucklin, even when there are three new alternatives.*

PROOF. We prove the NP-hardness of both PcWNA and PWNA by the same reduction from the special case of 3DM mentioned in Proposition 1. Given any 3DM instance where $|A| = |B| = |X| = q$, q is even, $t = 3q/2$, and no element in $A \cup B \cup X$ appears in more than 6 elements in T, we construct a PcWNA (PWNA) instance as follows. Without loss of generality, assume $q \geq 5$; otherwise the instance 3DM can be solved directly.

Alternatives: $A \cup B \cup X \cup Y \cup D \cup \{c\}$, where $Y = \{y_1, y_2, y_3\}$ is the set of new alternatives, and $D = \{d_1, \ldots, d_{9q^2}\}$ is the set of auxiliary alternatives.

Votes: For any $i \leq 2q + 1$, we define a vote V_i. Let $P = (V_1, \ldots, V_{2q+1})$. Instead of defining these votes explicitly, below we give the properties that P satisfies. The votes can be constructed in polynomial time.

(i) For any $i \leq q$, c is ranked in the first position. Suppose $S_i = (a, b, x)$. Then, let a, b, x be ranked in the $(3q + 1)$th, $(3q + 2)$th, and $(3q + 3)$th positions in V_i, respectively.

(ii) For any i such that $q < i \leq 3q/2 = t$, c is ranked in the $(3q + 4)$th position. Suppose $S_i = (a, b, x)$. Then, let a, b, x be ranked in the $(3q + 1)$th ,$(3q + 2)$th, and $(3q + 3)$th positions in V_i, respectively.

(iii) For any i such that $3q/2 < i \leq 2q + 1$, let c be ranked in the $(3q + 4)$th position, and no alternative in $A \cup B \cup X$ is ranked in the $(3q + 1)$th, $(3q + 2)$th, or $(3q + 3)$th position in V_i.

(iv) For any $c' \in A \cup B \cup X$, c' is ranked within top $3q + 3$ positions for exactly $q + 1$ times in P; and c' is never ranked in the $(3q + 4)$th position.

(v) For any $d \in D$, d is ranked within top $3q + 4$ positions at most once.

The existence of a profile P that satisfies (iv) is guaranteed by the assumption that in the 3DM instance, $q \geq 5$, no element is covered more than 6 times, and there are enough positions within top $3q+3$ positions in all votes to ensure that each alternatives in $\mathcal{C}$ appears exactly $q + 1$ times. We note that there are in total $9q^2$ auxiliary alternatives, and the total number of top $3q + 4$ positions in all votes is $(3q + 4)(2q + 1) < 9q^2$. Therefore, (v) can be satisfied. It follows that there exists a profile P that satisfies (i), (ii), (iii), (iv), and (v), and such a profile can be constructed in polynomial time (by first putting the alternatives to their positions defined in (i), (ii), and (iii), then filling out the positions using remaining alternatives to meet conditions (iv) and (v)). The Bucklin score of c is $3q+4$ in P. For any $j \leq q$, the Bucklin score of a_j (resp., b_j, x_j) is at most $3q+3$ in P, and for any $j \leq 9q^2$, the Bucklin score of $d_j \in D$ is at least $3q + 4$ in P. Observe that the Bucklin score of any alternative cannot be decreased in any extension of P.

Suppose that the 3DM instance has a solution, denoted by $\{S_j : j \in J\}$, where $J \subseteq \{1, \ldots, t\}$. For any $j \in J$, we let V'_j be the extension of V_j in which y_1, y_2, y_3 are ranked in the $(3q + 1)$th, $(3q + 2)$th, and $(3q + 3)$th positions, respectively. For any

$j \in \{1, \ldots, 2q+1\} \setminus J$, we let V'_j be the extension of V_j where $\{y_1, y_2, y_3\}$ are ranked in the bottom positions. Let $P' = (V'_1, \ldots, V'_{2q+1})$. It follows that in P', the Bucklin score of c is $3q+4$ and c is ranked within top $3q+4$ for $3q/2$ times; the Bucklin score of any other alternative is at least $3q+4$, and none of them is ranked within top $3q+4$ for more than $q+1$ times. Therefore, c is the unique winner for Bucklin for P', which means that there is a solution to the PcWNA (PWNA) instance.

Conversely, suppose that there is a solution to the PcWNA (PWNA) instance, denoted by $P' = (V'_1, \ldots, V'_{2q+1})$. We recall that in order for c to be a co-winner, the Bucklin score of any alternative in $A \cup B \cup X$ must be at least $3q+4$ (since the Bucklin score of c cannot decrease in P'). Therefore, for every $a \in A$, there exists $i \leq t$ such that a is ranked within top $3q+3$ positions in V_i, and is ranked lower than the $(3q+3)$th position in V'_i. Consequently, in each of such V'_i, the new alternatives must be ranked within top $3q+3$ positions. Because $|A| = q$, each new alternative must be ranked within top $3q+3$ positions in $V_1, \ldots, V_t$ for q times. Because c is a co-winner, no alternative in Y is ranked within top $3q+3$ positions in P' for more than q times. Therefore, in exactly q votes in P', the alternatives in Y are ranked within top $3q+3$ positions. Let $\{V'_{i_1}, \ldots, V'_{i_q}\}$ denote these votes.

We claim that $\{S_{i_1}, \ldots, S_{i_q}\}$ is a solution to the 3DM instance. If not, then there exists $e \in B \cup X$ that does not appear in any S_{i_j}. However, it follows that e is ranked within top $3q+3$ positions for exactly q times, which means that the Bucklin score of e is at most $3q+3$. Therefore, the Bucklin score of e is lower than the Bucklin score of c. This contradicts the assumption that c is a co-winner for P'. Therefore, the PcWNA (PWNA) problem is NP-hard for Bucklin. □

6. COPELAND$_0$

For any profile P, the Copeland score of an alternative $c \in \mathcal{C}$ in profile P is denoted by $\text{CS}_P(c) = |\{c' \in \mathcal{C} : N_P(c, c') > n/2\}|$ (recall that we focus on Copeland$_0$, which means that the tie in a pairwise election gives 0 point to both participating alternatives). We have the following straightforward observation.

Property 1 *For any profile P' over $\mathcal{C} \cup \{y\}$ that is an extension of profile P, the following inequalities hold:*

$$\forall c \in \mathcal{C},\ CS_P(c) \leq CS_{P'}(c) \leq CS_P(c) + 1 \tag{1}$$

We prove that a useful restriction of X3C remains NP-complete.

Proposition 2 X3C *is* NP*-complete, even if* $t = 2q-2$ *and* $\Delta(I) \leq 6$.

PROOF. The proof is similar to the proof for Proposition 1. Let $I = (\mathcal{S}, \mathcal{V})$ be an instance of X3C, where $\mathcal{V} = \{v_1, \ldots, v_{3q}\}$ and $\mathcal{S} = \{S_1, \ldots, S_t\}$. We next show how to build an instance $I' = (\mathcal{S}', \mathcal{V}')$ of X3C in polynomial time, with $|\mathcal{V}'| = 3q'$ and $|\mathcal{S}'| \leq 6$ such that $t' = 2q' - 2$ and $\Delta(I') \leq 6$.

• If $t < 2q-2$, then we add $2q-2-t$ dummy 3-sets to $\mathcal{S}$ by duplicating $2q-2-t$ sets of $\mathcal{S}$ once each. It follows from $t \geq q$ that $2q-2-t \leq q-2 < t$.

• If $t > 2q-2$, then we add $3(t-2q+2)$ new elements $v'_1, \ldots, v'_{3(t-2q+2)}$ and $t-2q+2$ 3-sets $\{v'_1, v'_2, v'_3\}, \ldots, \{v'_{3(t-2q+2)-2}, v'_{3(t-2q+2)-1}, v'_{3(t-2q+2)}\}$.

The size of the input of the new instance is polynomial in the size of the input of the old instance. Moreover, I is a yes-instance if and only if I' is also a yes-instance. Finally, in the new instance I', we have: $|\mathcal{V}'| = |\mathcal{V}| = 3q$ and $t' = |\mathcal{S}'| = t+(2q-2-t) = 2q-2 = 2q'-2$ in the first case, while $3q' = |X'| = 3q+3(t-2q+2) = 3(t-q+2)$ and $t' = |\mathcal{S}'| = t+(t-2q+2) = 2(t-q+1) = 2(q'-1)$ in the second case. Moreover, $d_{I'}(v) \leq 2d_I(v) \leq 6$ if $v \in \mathcal{V}$, and $d_{I'}(v) = 1$ if $v \in \mathcal{V}' \setminus \mathcal{V}$. □

Theorem 3 *The PcWNA problem is* NP*-complete for Copeland$_0$, even when there is one new alternative.*

PROOF. The proof is by a reduction from X3C. Let $I = (\mathcal{S}, \mathcal{V})$, where $t = 2q-2$ and $\Delta(I) \leq 6$ be an instance of X3C as described in Proposition 2. As previously, we can assume $q \geq 8$; hence $\Delta(I) \leq q-2$. For any X3C instance, we construct the following PcWNA instance for Copeland$_0$.

Alternatives: $\mathcal{V} \cup D \cup Y \cup \{c\}$, where $D = \{d_1, \ldots, d_t\}$ and $Y = \{y\}$ is the set of the new alternative.

Votes: For any $i \leq t$, we define the following $2t$ votes.

$$V_i = [d_i \succ (D \setminus \{d_i\}) \succ (\mathcal{V} \setminus S_i) \succ c \succ S_i]$$

$$V'_i = [\text{rev}(S_i) \succ \text{rev}(\mathcal{V} \setminus S_i) \succ \text{rev}(D \setminus \{d_i\}) \succ c \succ d_i]$$

Here the elements in a set are ranked according to the order of their subscripts, i.e., if $S_i = \{v_2, v_5, v_7\}$, then the elements are ranked as $v_2 \succ v_5 \succ v_7$. For any set X such that $X \subset \mathcal{V}$ or $X \subset D$, let $\text{rev}(X)$ denote the linear order where the elements in X are ranked according to the reversed order of their subscripts. For example, $\text{rev}(\{v_2, v_5, v_7\}) = v_7 \succ v_5 \succ v_2$.

We also define the following $t = 2q-2$ votes.

$$W_1 = \ldots = W_{q-1} = [\mathcal{V} \succ D \succ c]$$

$$W'_1 = \ldots = W'_{q-1} = [\text{rev}(D) \succ \text{rev}(\mathcal{V}) \succ c]$$

Let $P = (V_1, V'_1, \ldots, V_t, V'_t, W_1, W'_1, \ldots, W_{q-1}, W'_{q-1})$.

We note that there are $3t$ votes in the instance. We recall that by assumption, $3t/2 = 3q-3$. We make the following observations on the function N_P.

• *For any $d \in D$, d beats c*: this holds because $N_P(c, d) = 1$.

• *For any $v \in \mathcal{V}$, v beats c*: this holds because $N_P(c, v) = d_I(v) \leq q-2 < 3q-3$.

• *For any $d \in D$ and $v \in \mathcal{V}$, d and v are tied*: this holds because $N_P(v, d) = t+q-1 = 3q-3$.

• *For any $v, v' \in \mathcal{V}$ ($v' \neq v$), v and v' are tied.*

• *For any $d, d' \in D$ ($d' \neq d$), d and d' are tied.*

From these observations we have the following calculation on the Copeland scores:

• $\text{CS}_P(c) = 0$.

• For any $v \in \mathcal{V}$, $\text{CS}_P(v) = 1$.

• For any $d \in D$, $\text{CS}_P(d) = 1$.

Now, assume that $I = (\mathcal{S}, \mathcal{V})$ is a yes-instance of X3C; hence, there exists $J \subset \{1, \ldots, t\}$ with $|J| = q$ and $\bigcup_{j \in J} S_j = \mathcal{V}$. Next, we show how to make c a co-winner by introducing one new alternative y.

• For any $j \in J$, we let $\widetilde{V}_j = [d_j \succ D \setminus \{d_j\} \succ \mathcal{V} \setminus S_j \succ c \succ y \succ S_j]$ be the completion of V_j.

• For any $i \leq t$, we let $\widetilde{V}'_i = [\text{rev}(S_i) \succ \text{rev}(\mathcal{V} \setminus S_i) \succ \text{rev}(D \setminus \{d_i\}) \succ c \succ y \succ d_i]$ be the completion of V'_i.

• For any vote not mentioned above, we put y in the top position.

• Finally, let P' denote the profile obtained in the above way.

It follows that y loses to c in their pairwise election, and for any other alternative $c' \in \mathcal{C}$ ($c' \neq y$ and $c' \neq c$), c' and y are tied in their pairwise election. Therefore, the Copeland score is 1 for c, any alternative in $\mathcal{V}$, and any alternative in D; the Copeland score of y is 0. It follows that c is a co-winner.

Next, we show how to convert a solution to the PcWNA instance to a solution to the X3C instance. Let $P' = (\widetilde{V}_1, \ldots, \widetilde{V}_t, \widetilde{V}'_1, \ldots, \widetilde{V}'_t, \widetilde{W}_1, \widetilde{W}'_1, \ldots, \widetilde{W}_{q-1}, \widetilde{W}'_{q-1})$ be a profile with the new alternative, such that c becomes a co-winner according to the Copeland$_0$ rule.

We denote $P'_1 = (\widetilde{V}_1, \ldots, \widetilde{V}_t)$, $P'_2 = (\widetilde{V}'_1, \ldots, \widetilde{V}'_t)$ and $P'_3 = (\widetilde{W}_1, \widetilde{W}'_1, \ldots, \widetilde{W}_{q-1}, \widetilde{W}'_{q-1})$. It follows from the above observations on Copeland scores of alternatives in profile P and inequalities (1) of Property 1, that $\text{CS}_{P'}(c) = 1, \forall c' \in D \cup \mathcal{V}$, $\text{CS}_{P'}(c) = 1$ and $\text{CS}_{P'}(y) \leq 1$.

We now claim the following.

(a) $\forall v \in \mathcal{V}$, $N_{P'}(v, y) \leq 3q - 3$, $N_{P'}(y, c) = 3q - 2$ and $\forall d \in D$, $N_{P'}(d, y) = 3q - 3$. $N_{P'_2}(c, y) = t = 2q - 2$. Moreover, for any $i \leq t$, $c \succ y \succ d_i$ in $\widetilde{V}'_i$.

(b) $\forall v \in \mathcal{V}$, $N_{P'_2 \cup P'_3}(v, y) \geq N_{P'_2 \cup P'_3}(c, y)$.

For (a). Since c is a co-winner for P', c must beat y in their pairwise election. Meanwhile, any $c' \in \mathcal{V} \cup D$ cannot beat y in their pairwise elections. Therefore, we must have that $N_{P'}(c, y) \geq 3q - 2$, and for any $c' \in \mathcal{V} \cup D$, $N_{P'}(c', y) \leq 3q - 3$. For any $d_i \in D$, in profile P', we have that $d_i \succ c$ except in $\widetilde{V}'_i$, which means that $N_{P'}(d_i, y) \geq N_{P'}(c, y) - 1$ by transitivity in each vote. Hence, $3q - 3 \geq N_{P'}(d_i, y) \geq N_{P'}(c, y) - 1 \geq 3q - 3$, which means that $N_{P'}(d_i, y) = 3q - 3$ and $N_{P'}(c, y) = 3q - 2$. From these equalities, we deduce that $\forall d \in D$, $N_{P'}(d, y) = N_{P'}(c, y) - 1$ and then, for any $i \leq t$, we have that $c \succ y \succ d_i$ in $\widetilde{V}'_i$. It follows that $N_{P'_2}(c, y) = t = 2q - 2$.

For (b). For any $v \in \mathcal{V}$, because in any vote in $P'_2 \cup P'_3$ $v \succ c$, by transitivity we have $N_{P'_2 \cup P'_3}(v, y) \geq N_{P'_2 \cup P'_3}(c, y)$.

Let $J = \{j \leq t : c \succ y \text{ in } \widetilde{V}_j\}$. We will prove that $|J| = q$ and $\cup_{j \in J} S_j = \mathcal{V}$. First, note that $|J| \leq q$ because $|J| = N_{P'_1}(c, y) \leq N_{P'}(c, y) - N_{P'_2}(c, y) = q$ from item (a).

Now, for any $v \in \mathcal{V}$ let $J_v = \{j \leq t : y \succ v \text{ in } \widetilde{V}_j\}$. We claim: $\forall v \in \mathcal{V}$, $J \cap J_v \neq \emptyset$. Otherwise, there exists $v^* \in \mathcal{V}$ with $J \cap J_{v^*} = \emptyset$. This means that $c \succ y$ implies $v^* \succ y$ in votes in P'_1. Hence, $N_{P'_1}(v^*, y) \geq N_{P'_1}(c, y)$. By adding this inequality with the inequality in item (b) (let $v = v^*$), we obtain that $N_{P'}(v^*, y) \geq N_{P'}(c, y)$. Now, combining the inequalities in item (a), we have that $3q - 3 \geq N_{P'}(v^*, y) \geq N_{P'}(c, y) = 3q - 2$, which is a contradiction. Therefore, for all $v \in \mathcal{V}$, $J \cap J_v \neq \emptyset$. Finally, since $|\mathcal{V}| = 3q$, $|S_i| = 3$ and $|J| \leq q$, we deduce that $|J| = q$ and $J = \{j \leq t : c \succ y \succ S_j \text{ in } \widetilde{V}_j\}$. Also, because for all $v \in \mathcal{V}$, $J \cap J_v \neq \emptyset$, we have $\bigcup_{j \in J} S_j = \mathcal{V}$. In conclusion, $I = (\mathcal{S}, \mathcal{V})$ is a yes-instance of X3C. This completes the NP-hardness proof for the PcWNA problem for Copeland$_0$. □

7. MAXIMIN

To prove the NP-hardness of the PcWNA problem for Maximin, we first make the following observation, whose proof is straightforward.

Property 2 *Let P be a profile over $\mathcal{C}$, P' be a profile over $\mathcal{C} \cup \{y\}$ such that P' is an extension P. The following (in)equalities hold:*

(i) $\forall c \in \mathcal{C}$, $Sim_{P'}(c) = \min\{Sim_P(c), N_{P'}(c, y)\}$.

(ii) $\forall c \in \mathcal{C}$, $Sim_{P'}(c) \leq Sim_P(c)$.

Theorem 4 *PcWNA and PWNA problems are* NP*-complete for maximin, even when there is one new alternative.*

PROOF. We first prove the NP-hardness for the PcWNA problem by a reduction from X3C. Let $I = (\mathcal{S}, \mathcal{V})$ with $t = 2q - 2$ and $\Delta(I) \leq 6$ be an instance of X3C as described in Proposition 2. Without loss of generality, assume $q \geq 8$; in particular, we deduce $\Delta(I) \leq q - 2$. We define a PcWNA instance for maximin as follows:

Alternatives: $\mathcal{V} \cup \{c, d\} \cup \{y\}$, where y is the new alternative.

Votes: For any $i \leq t$, we define the following vote. $V_i = [(\mathcal{V} \setminus S_i) \succ d \succ c \succ S_i]$. Let $W_1 = \cdots = W_{q-1} = [c \succ \text{rev}(\mathcal{V}) \succ d]$ and $W_q = [\text{rev}(\mathcal{V}) \succ d \succ c]$. Let $P_1 = (V_1, \ldots, V_t)$, $P_2 = (W_1, \ldots, W_q)$, and $P = P_1 \cup P_2$.

We make the following observation on the maximin scores of the alternatives before y is added.

- $Sim_P(c) = q - 1$. Indeed, $N_P(c, d) = q - 1$ and $\forall v \in \mathcal{V}$, $N_P(c, v) = q - 1 + d_I(v) \geq q$.
- $Sim_P(d) \leq 6 \leq q - 2$. This is because for any $v \in \mathcal{V}$, v is covered by the 3-sets for no more than $q - 2$ times (the assumption of the input X3C instance), which means that in P_1, $d \succ v$ for at most $q - 2$ times, i.e., $N_P(d, v) = d_I(v) \leq 6 \leq q - 2$.
- For any $v \in \mathcal{V}$, $Sim_P(v) \geq q$. Actually, $N_P(v, d) > N_P(v, c) = t - d_I(v) + 1 \geq q$. For any $i < j \leq 3q$, $N_P(v_i, v_j) = N_{P_1}(v_i, v_j) \geq t - d_I(v) \geq 2q - 2 - (q - 2) = q$ and if $i > j$, $N_P(v_i, v_j) \geq N_{P_2}(v_i, v_j) = q$.

Now, suppose the X3C instance has a solution $J \subset \{1, \ldots, t\}$ with $|J| = q$ and $\bigcup_{j \in J} S_j = \mathcal{V}$. We show how to make c a co-winner by introducing one new alternative y.

- For any $j \in J$, we let $V'_j = [(\mathcal{V} \setminus S_j) \succ d \succ c \succ y \succ S_j]$.
- For any $j \in \{1, \ldots, t\} \setminus J$, we let $V'_j = [y \succ (\mathcal{V} \setminus S_j) \succ d \succ c \succ S_j]$.
- For any $j \leq q - 1$, we let $W'_j = [c \succ y \succ \text{rev}(\mathcal{V}) \succ d]$.
- Let $W'_q = [y \succ \text{rev}(\mathcal{V}) \succ d \succ c]$.
- Finally, let $P' = (V'_1, \ldots, V'_t, W'_1, \ldots, W'_q)$.

In P', the maximin score of y is $q - 1$ (via c), because $t = 2q - 2$, which means that $t - q + 1 = q - 1$; the maximin score of c is $q - 1$ (via d); the maximin score of d is no more than $q - 1$ (via any of $v \in \mathcal{V}$); and the maximin score of any $v \in \mathcal{V}$ is $q - 1$ (via y). Therefore, c is a co-winner for the maximin rule.

Next, we show how to convert a solution P' to the above PcWNA instance for the maximin rule to a solution to the X3C instance. Let $P' = (V'_1, \ldots, V'_t, W'_1, \ldots, W'_q)$ be an extension of P with one new alternative y, and c is the maximin winner for P'. Let $P'_1 = (V'_1, \ldots, V'_t)$ and $P'_2 = (W'_1, \ldots, W'_q)$.

We make the following observations.

(a) $\forall v \in \mathcal{V}$, $N_{P'}(v, y) \leq q - 1$,

(b) $N_{P'}(y, c) \leq q - 1$ and $N_{P'}(y, d) \geq q$,

(c) $y \succ c$ in W'_q.

For item (a): Because c is a co-winner, for any $v \in \mathcal{V}$, $Sim_{P'}(v) \leq Sim_{P'}(c)$. We recall that $Sim_P(c) = q - 1$ and $Sim_P(v) \geq q$. Thus, by Property 2 we have the following calculation.

$$\min\{N_{P'}(v, y), q\} \leq Sim_{P'}(v) \leq Sim_{P'}(c) \leq Sim_P(c) = q - 1$$

For item (b): First from (a), we deduce that for any $v \in \mathcal{V}$, $N_{P'}(y, v) \geq t + q - N_{P'}(v, y) > q$. Thus, we obtain:

$$\begin{aligned} Sim_{P'}(y) &= \min\{N_{P'}(y, c), N_{P'}(y, d)\} \\ &\leq Sim_{P'}(c) \leq Sim_P(c) = q - 1 \end{aligned} \tag{2}$$

Now, assume $N_{P'}(y, d) \leq q - 1$. Then, $N_{P'_2}(d, y) = q - N_{P'_2}(y, d) \geq q - N_{P'}(y, d) \geq 1$. Hence, there exists $i \leq q$ such that in W'_i, we have that for any $v \in \mathcal{V}$, $v \succ d \succ y$. Moreover, $N_{P'_1}(d, y) = t - N_{P'_1}(y, d) \geq 2q - 2 - (q - 1) = q - 1$. Let $J_0 \subseteq \{1, \ldots, t\}$ (with $|J_0| = q - 1$) be the subscripts of arbitrary $q - 1$ votes in P'_1, where $d \succ y$. Because $|\mathcal{V}| = 3q$ and $|S_j| = 3$, there exists $v^* \in \mathcal{V} \setminus \bigcup_{j \in J_0} S_j$. We deduce that for all $j \in J_0$, $v^* \succ y$ in V'_j. In conclusion, $N_{P'}(v^*, y) \geq |J_0| + 1 = q$, which contradicts item (a). Using inequality (2), item (b) follows.

For item (c): Otherwise, by the definition of W_q, we deduce:

$$\forall v \in \mathcal{V}, \; N_{P'_2}(v, y) \geq 1 \tag{3}$$

On the other hand, using $N_{P'_1}(y, c) \leq N_{P'}(y, c)$ and item (b), we have $N_{P'_1}(c, y) = t - N_{P'_1}(y, c) \geq t - N_{P'}(y, c) \geq t - (q -$

$1) = q-1$. Let $J_0 \subseteq \{1,\ldots,t\}$ (with $|J_0| = q-1$) be the subscripts of arbitrary $q-1$ votes in P'_1, where $c \succ y$. We have $\mathcal{V} \setminus \bigcup_{j\in J_0} S_j \neq \emptyset$ since $|\mathcal{V}| = 3q$ and $|S_i| = 3$. Hence, there exists $v^* \in \mathcal{V} \setminus \bigcup_{j\in J_0} S_j$ such that:

$$N_{P'_1}(v^*, y) \geq |J_0| = q-1 \tag{4}$$

Summing up inequalities (3) (let $v = v^*$) and (4), we reach a contradiction with item (a).

From items (b) and (c), we get

$$N_{P'_1}(y,c) = N_{P'}(y,c) - N_{P'_2}(y,c) \leq q-1-1 = q-2$$

Thus, $N_{P'_1}(c,y) = t - N_{P'_1}(y,c) \geq t-(q-2) = q$. Let J denote the subscripts of arbitrary q votes in P'_1 where $c \succ y$. We claim $\bigcup_{j\in J} S_j = \mathcal{V}$. Otherwise, there exists $v^* \in \mathcal{V} \setminus \bigcup_{j\in J} S_j$. It follows that for any $j \in J$, $v^* \in (\mathcal{V} \setminus \bigcup_{j\in J} S_j) \subseteq \mathcal{V} \setminus S_j$, which means that $v^* \succ c \succ y$ in V_j. Hence, $N_{P'}(v^*, y) \geq N_{P'_1}(v^*, y) \geq |J| = q$, which contradicts item (a). In conclusion, $I = (\mathcal{S}, \mathcal{V})$ is a yes-instance of X3C. Therefore, PcWNA is NP-complete for maximin.

For the PWNA problem, we make the following change. Let $W_q = [\text{rev}(\mathcal{V}) \succ c \succ d]$. Then, before the new alternative is introduced, the maximin score of c is q. Then, similarly we can prove the NP-hardness of the PWNA problem. □

8. PLURALITY WITH RUNOFF

In this section, we adopt the parallel-universe tie-breaking. If a tie occurs in the first round, then all possible compatible second rounds are considered: for instance, if the plurality scores, ranked in decreasing order, are $x_1 \mapsto 8, x_2 \mapsto 6, x_3 \mapsto 6, x_4 \mapsto 5 \ldots$, then the set of co-winners contains the majority winner between x_1 and x_2 and the majority winner between x_1 and x_3. We show a necessary and sufficient condition for a given alternative c to be a possible co-winner with new alternatives for plurality with runoff. This condition can be easily converted to a polynomial-time algorithm that computes PcWNA for plurality with runoff. For any profile P and any alternative $x \in \mathcal{C}$, we let $S^P(x)$ denote the plurality score of x in P, that is, the number of times where x is ranked in the first position in votes in P. We let $X_P^-(c)$ denote the set of alternatives that lose to c in their pairwise elections, and let $X_P^+(c) = \mathcal{C} \setminus (X_P^-(c) \cup \{c\})$.

Proposition 3 *For any profile P and any alternative c, c is a possible co-winner with k new alternatives under P for plurality with runoff, if and only if one of the two following conditions holds:*

1. *there exists an alternative $d \in X_P^-(c)$ such that $\sum_{x\in\mathcal{C}\setminus\{c,d\}} \max(0, S^P(x) - \theta) \leq k\theta$, where $\theta = \min\left(S^P(d), S^P(c)\right)$.*

2. $\sum_{x\in\mathcal{C}\setminus\{c\}} \max(0, S^P(x) - S^P(c)) \leq \lfloor n/2 \rfloor + (k-1)S^P(c)$.

PROOF. Let $P = (V_1,\ldots,V_n)$ be a profile over $\mathcal{C}$ and $P' = (V'_1,\ldots,V'_n)$ be a completion of P with k new alternatives. c is a co-winner in P' if one of the following conditions hold:

1. c and $d \in \mathcal{C} \setminus \{c\}$ are possible second round competitors, and c (weakly) beats d in their pairwise election under P'.

2. c and $y \in Y$ are possible second round competitors, and c (weakly) beat y in their pairwise election under P'.

Let $\succeq_M^P$ denote a weak majority relations under P, defined as follows. For any pair of alternatives a, b, $a \succeq_M^P b$ if at least half of the voters in P prefers a to b. $\succeq_M^{P'}$ is defined similarly. Let us first analyze the situations in which 1 occurs. First, in order to have $c \succeq_M^{P'} d$ we must have $c \succeq_M^P d$ (because the relative positions of c and d are the same in V_i and V'_i). Thus, 1 occurs if and only there exists an alternative d that loses to c in their pairwise elections and such that c and d can compete in the second round. Fix such d. In order for c and d to be possible second round competitors, we must have $\min(S^{P'}(c), S^{P'}(d)) \geq S^{P'}(x)$ for every $x \in \mathcal{C}\setminus\{c,d\} \cup Y$. Without loss of generality, we can assume that the scores of c and d are the same in P and P', and similarly for the scores of any $x \in \mathcal{C}$ such that $S^P(\mathcal{C}) \leq \min(S^P(c), S^P(d))$, since these alternatives do not need to lose any point to allow a possible second round between c and d. Let $\hat{\mathcal{C}}_{c,d}$ be the set of all candidates x in $\mathcal{C} \setminus \{c,d\}$ such that $S^P(x) > \min(S^P(c), S^P(d))$. Each candidate $x \in \hat{\mathcal{C}}_{c,d}$ has to lose at least $S^P(x) - \min(S^P(c), S^P(d))$ points, and for this we need $\sum_{x\in\hat{\mathcal{C}}_{c,d}} S^P(x) - \min(S^P(c), S^P(d))$ points to be given to the new candidates. Therefore, to have c and d (possibly) in the second round, the number of points we must distribute to new candidates is $\sigma = \sum_{x\in\mathcal{C}\setminus\{c,d\}} \max(0, S^P(x) - \theta)$, where $\theta = \min(S^P(c), S^P(d))$. Now, we also need the score of any new alternative y to be at most θ, therefore we need $\sigma \leq k\theta$. This leads to the condition 1 in the statement of the proposition.

Now, let us analyze the conditions allowing condition 2 to occur. In order to have c in the second round and none of the alternatives in $\mathcal{C} \setminus \{c\}$ enter the second round, we need to distribute $\kappa = \sum_{x\in\mathcal{C}\setminus\{c\}} \max(0, S^P(x) - \theta)$ points to the candidates in $\mathcal{C}$. Let y^* be the new alternative that enters the second round together with c. y^* can take at most $\lfloor n/2 \rfloor$ points, otherwise y^* will beat c in their pairwise election. For any other new alternative y' can take at most $S^P(c)$ points. Therefore, we must have that

$$\kappa \leq \lfloor n/2 \rfloor + (k-1)S^P(c)$$

It is straightforward that if the above equation holds, then there exists a way to extend P to P' with k new alternatives such that c is the winner for plurality with runoff. This leads to condition 2 in the statement of the proposition.

Therefore, c is a PcWNA if and only if one of the two conditions in the statement of the proposition holds. □

Example 1 *Let P be the following 4-candidate, 18-voter profile: 4 votes of $a \succ b \succ c \succ d$, 3 votes of $b \succ a \succ c \succ d$, 7 votes of $d \succ a \succ c \succ b$, 2 votes of $d \succ c \succ b \succ a$ and 2 votes of $c \succ a \succ b \succ d$. We want to determine if c is a possible co-winner with k new alternatives for plurality with runoff. Note that $X_P^-(c) = \{b, d\}$. For condition 1 to be satisfied, it suffices to consider d as the competitor for c. Then, $\theta = 2$ and condition 1 is satisfied if* $3 \leq 2k$, i.e., $k \geq 2$. *For condition 2 to be satisfied, we have $\sum_{x\in\mathcal{C}\setminus\{c\}} \max(0, S^P(x) - S^P(c)) = 10$, $\lfloor n/2 \rfloor = 9$. Therefore, condition 2 is satisfied if and only if $k \geq 2$. It follows that as soon as we have at least two new candidates, c is a possible co-winner.*

We also obtain a similar proposition for PWNA, whose proof is similar to the proof of Proposition 3, therefore is omitted.

Proposition 4 *For any profile P and any alternative c, c is a possible winner with k new alternatives under P for plurality with runoff, if and only if one of the three following conditions holds:*

1. there exists an alternative $d \in X_P^-(c)$ such that $S^P(d) \geq S^P(c)$ and $\sum_{x\in\mathcal{C}\setminus\{c,d\}} \max(0, S^P(x) - S^P(c) + 1) \leq k(S^P(c) - 1)$;

2. there exists an alternative $d \in X_P^-(c)$ such that $S^P(d) < S^P(c)$ and $\sum_{x\in X_P^-(c)\setminus\{d\}} \max(0, S^P(x) - S^P(d))$ $+ \sum_{x\in X_P^+(c)} \max(0, S^P(x) - S^P(d) + 1) \leq kS^P(d)$;

3. $\sum_{x \in \mathcal{C} \setminus \{c\}} \max(0, S^P(x) - S^P(c) + 1) \leq \lfloor (n+1)/2 \rfloor + (k-1)(S^P(c) - 1)$.

Corollary 1 *Determining whether $c \in \mathcal{C}$ is a possible (co-)winner for plurality with runoff is in* P.

9. CONCLUSION

In this paper we have gone beyond existing results on the complexity of the possible (co-)winner problem with new alternatives. While [6, 7] focused on scoring rules, we have identified three new rules for which the PcWNA problem is NP-complete (Bucklin, Copeland, and maximin). We also showed that the PcWNA problem has a polynomial time algorithm for plurality with runoff, and as far as approval voting is concerned, we examined three definitions of the extension of a profile to new alternatives and showed that depending on which definition we chose, the problem can be trivial or NP-complete. Our NP-completeness proofs and algorithms for the PcWNA problems, except for Copeland$_0$, can be extended to the PWNA problems for approval, Bucklin, maximin, and plurality with runoff. The results are summarized in the following table. These results can be compared with results for control by adding candidates and cloning. Control by adding candidates is NP-complete for most of voting rules considered here, namely Copeland [14], maximin [12], Borda, plurality with runoff and l-approval for $l \geq 2$ [9]; on the other hand, approval voting is immune to control by adding candidates [17]. Manipulability by cloning with positive probability (0-cloning) is polynomial for Borda, maximin and plurality with runoff, and NP-complete for Copeland and l-approval for $l \geq 2$ [9]. This shows that P(c)WNA, when viewed as a control problem, shows a resistance to strategic behaviour globally stronger than cloning and weaker than control by adding candidates.

Voting rule	PcWNA	PWNA
Borda	P [7]	
2-approval	P [7]	
l-approval, $l \geq 3$	NP-complete [2] [7]	
Approval	P (Def. 1) NP-complete (Def. 2) Trivial (Def. 3)	
Bucklin	NP-complete [2]	
Copeland$_0$	NP-complete [3]	?
maximin	NP-complete [3]	
Plurality with runoff	P	

Table 1: Complexity of PcWNA and PWNA problems for some common voting rules.

An obvious and interesting direction for future research is studying the computational complexity of the PcWNA (PWNA) problems for more common voting rules, including STV, Copeland$_\alpha$ (for some $\alpha \neq 0$), ranked pairs, and voting trees. Even for Copeland$_0$, the complexity of the PWNA problem still remains open. Moreover, viewing P(c)WNA problem as a control problem where the chair can add new candidates but do not know the preferences of the voters over the new candidates, it is interesting to know which voting rules are more resistant to this type of control from a non-computational viewpoint.

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[2]Even with 3 new alternatives.

[3]Even with 1 new alternative.

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