The Complexity of Voter Partition in Bucklin and Fallback Voting: Solving Three Open Problems

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ABSTRACT

Electoral control models ways of changing the outcome of an election via such actions as adding/deleting/partitioning either candidates or voters. These actions modify an election's participation structure and aim at either making a favorite candidate win ("constructive control") or prevent a despised candidate from winning ("destructive control"). To protect elections from such control attempts, computational complexity has been used to show that electoral control, though not impossible, is computationally prohibitive. Recently, Erdélyi and Rothe [10] proved that Brams and Sanver's fallback voting [5], a hybrid voting system that combines Bucklin with approval voting, is resistant to each of the standard types of control except five types of voter control. They proved that fallback voting is vulnerable to two of those control types, leaving the other three cases open.

We solve these three open problems, thus showing that fallback voting is resistant to all standard types of control by partition of voters—which is a particularly important and well-motivated control type, as it models "two-district gerrymandering." Hence, fallback voting is not only fully resistant to candidate control [10] but also fully resistant to constructive control, and it displays the broadest resistance to control currently known to hold among natural voting systems with a polynomial-time winner problem. We also show that Bucklin voting behaves almost as good in terms of control resistance. Each resistance for Bucklin voting strengthens the corresponding control resistance for fallback voting.

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity;
I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence—*Multiagent Systems*

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Economics, Theory

Keywords

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1. INTRODUCTION

Elections have been used for preference aggregation not only in the context of politics and human societies, but also in artificial intelligence, especially in multiagent systems, and other topics in computer science (see, e.g., [8, 14, 7]). That is why it is important to study the computational properties of voting systems. In particular, complexity can be used to protect elections against tampering attempts in control, manipulation, and bribery attacks by showing that such attacks, though not impossible in principle, can be computationally prohibitive.

Since the seminal paper of Bartholdi et al. [2], the complexity of *electoral control*—changing the outcome of an election via such actions as adding/deleting/partitioning either candidates or voters—has been studied for a variety of voting systems. Unlike *manipulation* [1, 6], which models attempts of strategic voters to influence the outcome of an election via casting insincere votes, control models ways of an external actor, the "chair," to tamper with an election's participation structure so as to alter its outcome. Another way of tampering with the outcome of elections is *bribery* [11, 12], which shares with manipulation the feature that votes are being changed, and with control the aspect that an external actor tries to change the outcome of the election. For more background on complexity results for control, manipulation, and bribery in approval voting and its variants, we refer to the survey of Baumeister et al. [3].

Regarding control, a central question is to find voting systems that are computationally resistant to as many of the common 22 control types as possible, where resistance means the corresponding control problem is NP-hard. Each control type is either constructive (the chair seeking to make some candidate win) or destructive (the chair seeking to make some candidate end up not winning). Erdélyi and Rothe [10] recently proved that fallback voting [5], a hybrid voting system combining Bucklin with approval voting, is resistant to each of these 22 standard control types except five types of voter control. They proved that fallback voting is vulnerable to two of those control types (i.e., these control problems are polynomial-time solvable), leaving the other three cases open. We solve these three open problems by showing that fallback voting is resistant to constructive and destructive control by partition of voters in the tie-handling model "ties promote" and to destructive control by partition of voters in the "ties eliminate" model. Partition of voters is a particularly important and well-motivated control type, as it models "two-district gerrymandering." Control-by-partition cases are the most difficult control types to deal with; their resistance proofs require the most involved constructions.

Thus fallback voting is fully resistant not only to candidate control [10] but also to constructive control. In terms of the total number of proven resistances it even outnumbers "sincere-strategy preference-based approval voting" (SP-AV, a modification [9] of another hybrid system proposed by Brams and Sanver [4]): Fallback voting has the most (20 out of 22) proven resistances to control among natural voting systems with a polynomial-time winner problem. Among such systems, only SP-AV (with its 19 proven control resistances [9]) and plurality voting were previously known to be fully resistant to candidate control [2, 15], and only Copeland voting and SP-AV were previously known to be fully resistant to constructive control [12, 9]. However, plurality has fewer resistances to voter control, Copeland voting has fewer resistances to destructive control, and SP-AV is missing one destructive voter partition resistance and—perhaps more importantly—is arguably less natural a system than fallback voting, since in SP-AV (as modified by Erdélyi and Rothe [9]) it may happen that votes are rewritten to ensure admissibility (for further details see [3, 9]).

We also study the control complexity of Bucklin voting itself and show that it has (at least) 19 resistances to control, thus drawing level with SP-AV. In particular, also Bucklin voting is—like SP-AV and fallback voting—fully resistant to constructive control and to candidate control. Since Bucklin voting is a special case of fallback voting, each resistance result for Bucklin strengthens the corresponding resistance result for fallback voting.

2. PRELIMINARIES

Elections and Voting Systems.

An *election* (C,V) is given by a finite set C of candidates and a finite list V of votes over C. A *voting system* is a rule that specifies how to determine the winner(s) of any given election. The two voting systems considered in this paper are Bucklin voting and fallback voting.

In *Bucklin voting*, votes are represented as linear orders over C, i.e., each voter ranks all candidates according to his or her preferences. For example, if $C = \{a,b,c,d\}$ then a vote might look like $c\ d\ a\ b$, i.e., this voter (strictly) prefers c to d, d to a, and a to b. Given an election (C,V) and a candidate $c \in C$, define the *level i score of c in* (C,V) (denoted by $score^i_{(C,V)}(c)$) as the number of votes in V that rank c among their top i positions. Denoting the *strict majority threshold for a list V of voters* by $maj(V) = \lfloor \|V\|/2 \rfloor + 1$, the *Bucklin score of c in* (C,V) is the smallest i such that $score^i_{(C,V)}(c) \geq maj(V)$. All candidates with a smallest Bucklin score, say k, and a largest level k score are the *Bucklin winners (BV winners, for short) in* (C,V). If some candidate becomes a Bucklin winner on level k, we call him or her a *level k BV winner in* (C,V). Note that a level 1 BV winner must be unique, but there may be more level k BV winners than one for $k > 1$, i.e., an election may have more than one Bucklin winner in general.

Brams and Sanver [5] proposed fallback voting as a hybrid voting system that combines Bucklin with approval voting. In *approval voting*, votes are represented by approval vectors in $\{0,1\}^{\|C\|}$ (with respect to a fixed order of the candidates in C), where 0 stands for disapproval and 1 stands for approval. Given an election (C,V) and a candidate $c \in C$, define the *approval score of c in* (C,V) (denoted by $score_{(C,V)}(c)$) as the number of c's approvals in (C,V), and all candidates with a largest approval score are the *approval winners in* (C,V). Note that an election may have more than one approval winner. *Fallback voting* combines Bucklin with approval voting as follows. Each voter provides both an approval vector and a linear ordering of all approved candidates. For simplicity, we will omit the disapproved candidates in each vote. For example, if $C = \{a,b,c,d\}$ and a voter approves of a, c, and d but disapproves of b, and prefers c to d and d to a, then this vote will be written as: $c\ d\ a$. We will always explicitly state the candidate set, so it will always be clear which candidates participate in an election and which of them are disapproved by which voter (namely those not occurring in his or her vote). Given an election (C,V) and a candidate $c \in C$, the notions of *level i score of c in* (C,V) and *level k fallback voting winner (level k FV winner, for short) in* (C,V) are defined analogously to the case of Bucklin voting, and if there exists a level k FV winner for some $k \leq \|C\|$, he or she is called a *fallback winner (FV winner, for short) in* (C,V). However, unlike in Bucklin voting, in fallback voting it may happen that no candidate reaches a strict majority for any level, due to voters being allowed to disapprove of (any number of) candidates, so it may happen that for no $k \leq \|C\|$ a level k FV winner exists. In such a case, every candidate with a largest (approval) score is an *FV winner in* (C,V). Note that Bucklin voting is the special case of fallback voting where each voter approves of all candidates. As a notation, when a vote contains a subset of the candidate set, such as $c\ D\ a$ for a subset $D \subseteq C$, this is a shorthand for $c\ d_1\ \cdots\ d_\ell\ a$, where the elements of $D = \{d_1, \ldots, d_\ell\}$ are ranked with respect to some (tacitly assumed) fixed ordering of all candidates in C. For example, if $C = \{a,b,c,d\}$ is assumed to be ordered lexicographically and $D = \{b,d\}$ then "$c\ D\ a$" is a shorthand for $c\ b\ d\ a$.

Types of Electoral Control.

There are eleven types of electoral control, each coming in two variants. In *constructive control* [2], the chair tries to make his or her favorite candidate win; in *destructive control* [15], the chair tries to prevent a despised candidate's victory. We refrain from giving a detailed discussion of natural, real-life scenarios for each of these 22 standard control types that motivate them; these can be found in, e.g., [2, 15, 12, 16, 9, 3]. However, we stress that every control type is motivated by an appropriate real-life scenario.

When we define our 22 standard control types as decision problems, we assume that each election or subelection in these control problems will be conducted with the voting system at hand (i.e., either Bucklin or fallback voting) and that each vote will be represented as required by the corresponding voting system. We also assume that the chair has complete knowledge of the voters' preferences and/or approval strategies. This assumption may be considered to be unrealistic in certain settings, but is reasonable and natural in certain others, including small-scale elections among humans and even large-scale elections among software agents. More to the point, assuming the chair to have complete information makes sense for our results, as most of our results are NP-hardness lower bounds showing resistance of a voting system against specific control attempts and complexity lower bounds in the complete-information model are inherited by any natural partial-information model (see [15] for a more detailed discussion of this point).

All our decision problems are formally described in the standard Instance-Question format. As an explicit example, we define the decision problem corresponding to control by partition of voters with the tie-handling rule "ties promote" (TP), see [15]. This control type produces a two-stage election with two first-stage and one final-stage subelections. The constructive variant of this problem is defined as:

Constructive Control by Partition of Voters (TP)

Instance: A set C of candidates, a list V of votes over C, and a designated candidate $c \in C$.

Question: Can V be partitioned into V_1 and V_2 such that c is the unique winner of the two-stage election in which the winners of the two first-stage subelections, (C, V_1) and (C, V_2), run against each other in the final stage?

The destructive variant of this problem is defined analogously, except it asks whether c is *not* a unique winner of this two-stage election. In both variants, if one uses the tie-handling model TE ("ties eliminate," see [15]) instead of TP in the two first-stage subelections, a winner w of (C, V_1) or (C, V_2) proceeds to the final stage if and only if w is the only winner of his or her subelection. Each of the four problems just defined can be seen as a way of modeling "two-district gerrymandering."

There are many ways of introducing new voters into an election—think, e.g., of "get-out-the-vote" drives, or of lowering the age-limit for the right to vote, or of attracting new voters with certain promises or even small gifts), and such scenarios are modeled as Constructive/Destructive Control by Adding Voters: Given a set C of candidates, two disjoint lists of votes over C (one list, V, corresponding to the already registered voters and the other list, W, corresponding to the as yet unregistered voters whose votes may be added), a designated candidate $c \in C$, and a nonnegative integer k, is there a subset $W' \subseteq W$ such that $\|W'\| \leq k$ and c is (is not) the unique winner in $(C, V \cup W')$?

Disenfranchisement and other means of voter suppression is modeled as Constructive/Destructive Control by Deleting Voters: Given a set C of candidates, a list V of votes over C, a designated candidate $c \in C$, and a nonnegative integer k, can one make c the unique winner (not a unique winner) of the election resulting from deleting at most k votes from V?

Having defined these eight standard types of voter control, we now turn to the 14 types of candidate control. Now, the control action seeks to influence the outcome of an election by either adding, deleting, or partitioning the candidates, again for both the constructive and the destructive variant.

In the adding candidates cases, we distinguish between adding, from a given pool of spoiler candidates, an *unlimited* number of such candidates (as originally defined by Bartholdi et al. [2]) and adding a *limited* number of spoiler candidates (as defined by Faliszewski et al. [12], to stay in sync with the problem format of control by deleting candidates and by adding/deleting voters). Constructive/Destructive Control by Adding (a Limited Number of) Candidates, is defined as follows: Given two disjoint candidate sets, C and D, a list V of votes over $C \cup D$, a designated candidate $c \in C$, and a nonnegative integer k, can one find a subset $D' \subseteq D$ such that $\|D'\| \leq k$ and c is (is not) the unique winner in $(C \cup D', V)$? The "unlimited" version of the problem is the same, except that the addition limit k and the requirement "$\|D'\| \leq k$" are being dropped, so *any* subset of the spoiler candidates may be added.

Constructive/Destructive Control by Deleting Candidates is defined by: Given a set C of candidates, a list V of votes over C, a designated candidate $c \in C$, and a nonnegative integer k, can one make c the unique winner (not a unique winner) of the election resulting from deleting at most k candidates (other than c in the destructive case) from C?

Finally, we define the partition-of-candidate cases, again using either of the two tie-handling models, TP and TE, but now we define these scenarios with and without a run-off. The variant with run-off, Constructive/Destructive Control by Run-Off Partition of Candidates, is analogous to the partition-of-voters control type: Given a set C of candidates, a list V of votes over C, and a designated candidate $c \in C$, can C be partitioned into C_1 and C_2 such that c is (is not) the unique winner of the two-stage election in which the winners of the two first-stage subelections, (C_1, V) and (C_2, V), who survive the tie-handling rule run against each other in the final stage? The variant without run-off is the same, except that the winners of first-stage subelection (C_1, V) who survive the tie-handling rule run against all members of C_2 in the final round (and not only against the winners of (C_2, V) surviving the tie-handling rule). As an example, think of a sports tournament in which certain teams (such as last year's champion and this year's hosting team) are given an exemption from qualification.

Immunity, Susceptibility, Resistance, Vulnerability.

Let $\mathfrak{CT}$ be a control type. We say a voting system is *immune to* $\mathfrak{CT}$ if it is impossible for the chair to make the given candidate the unique winner in the constructive case (not a unique winner in the destructive case) via exerting control of type $\mathfrak{CT}$. We say a voting system is *susceptible to* $\mathfrak{CT}$ if it is not immune to $\mathfrak{CT}$. A voting system that is susceptible to $\mathfrak{CT}$ is said to be *vulnerable to* $\mathfrak{CT}$ if the control problem corresponding to $\mathfrak{CT}$ can be solved in polynomial time, and is said to be *resistant to* $\mathfrak{CT}$ if the control problem corresponding to $\mathfrak{CT}$ is NP-hard. These notions are due to Bartholdi et al. [2] (except that we follow the now more common approach of Hemaspaandra et al. [16] who define *resistant* to mean "susceptible and NP-hard" rather than "susceptible and NP-complete").

Fallback voting is susceptible to each of our 22 control types [10]. It is easy to see that the same holds true for Bucklin voting. The proof is omitted.

Lemma 2.1. *Bucklin voting is susceptible to each of the 22 control types defined in this section.*

3. PARTITION OF VOTERS IN BV AND FV

Table 1 shows in boldface our results on the control complexity of fallback voting for three cases of voter partition (the other results for fallback voting being due to Erdélyi and Rothe [10]) and of Bucklin voting for all 22 standard control types. For comparison, this table also shows the results for approval voting due to Hemaspaandra et al. [15], and for SP-AV due to Erdélyi et al. [9].

In this section, we solve the three questions left open in [10]. We start with the proof that fallback voting is resistant to constructive control by partition of voters in model TP (see Corollary 3.2). We do so by proving in Theorem 3.1 that even Bucklin voting is resistant to this type of

Control by	Fallback Voting		Bucklin Voting		SP-AV		Approval	
	Const.	Dest.	Const.	Dest.	Const.	Dest.	Const.	Dest.
Adding Candidates (unlimited)	R	R	**R**	**R**	R	R	I	V
Adding Candidates (limited)	R	R	**R**	**R**	R	R	I	V
Deleting Candidates	R	R	**R**	**R**	R	R	V	I
Partition of Candidates	TE: R TP: R	TE: R TP: R	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: R TP: R	TE: R TP: R	TE: V TP: I	TE: I TP: I
Run-off Partition of Candidates	TE: R TP: R	TE: R TP: R	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: R TP: R	TE: R TP: R	TE: V TP: I	TE: I TP: I
Adding Voters	R	V	**R**	**V**	R	V	R	V
Deleting Voters	R	V	**R**	**V**	R	V	R	V
Partition of Voters	TE: R **TP: R**	**TE: R** **TP: R**	**TE: R** **TP: R**	**TE: R** **TP: S**	TE: R TP: R	TE: V TP: R	TE: R TP: R	TE: V TP: V

Table 1: Overview of results. Key: I = immune, S = susceptible, R = resistant, V = vulnerable, TE = ties eliminate, and TP = ties promote. Results new to this paper are in boldface.

control. As our reduction works also for the TE tie-handling model, this strengthens the corresponding result for fallback voting from [10].

Our reductions in the proof of Theorem 3.1 are from the NP-complete problem Exact Cover by Three-Sets, which is defined as follows (see, e.g., [13]):

Exact Cover by Three-Sets (X3C)

Instance: A set $B = \{b_1, b_2, \dots, b_{3m}\}$, $m \geq 1$, and a collection $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.

Question: Is there a subcollection $\mathcal{S}' \subseteq \mathcal{S}$ such that each element of B occurs in exactly one set in $\mathcal{S}'$?

Theorem 3.1. *Bucklin voting is resistant to constructive control by partition of voters in both model TE and model TP.*

Proof. Susceptibility holds by Lemma 2.1. To show NP-hardness we reduce X3C to our control problems. Let $(B, \mathcal{S})$ be an X3C instance with $B = \{b_1, b_2, \dots, b_{3m}\}$, $m \geq 1$, and a collection $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$. We define the election (C, V), where $C = B \cup \{c, w, x\} \cup D \cup E \cup F \cup G$ is the set of candidates with $D = \{d_1, \dots, d_{3nm}\}$, $E = \{e_1, \dots, e_{(3m-1)(m+1)}\}$, $F = \{f_1, \dots, f_{(3m+1)(m-1)}\}$, and $G = \{g_1, \dots, g_{n(3m-3)}\}$, and where w is the distinguished candidate. Let V consist of the following $2n + 2m$ voters:

1. For each $i, 1 \leq i \leq n$, there is one voter of the form: $c \;\; S_i \;\; G_i \;\; (G - G_i) \;\; F \;\; D \;\; E \;\; (B - S_i) \;\; w \;\; x$, where $G_i = \{g_{(i-1)(3m-3)+1}, \dots, g_{i(3m-3)}\}$ for each i, $1 \leq i \leq n$.
2. For each $i, 1 \leq i \leq n$, there is one voter of the form: $B_i \;\; D_i \;\; w \;\; G \;\; E \;\; (D - D_i) \;\; F \;\; (B - B_i) \;\; c \;\; x$, where, letting $\ell_j = \|\{S_i \in \mathcal{S} \mid b_j \in S_i\}\|$ for each j, $1 \leq j \leq 3m$, we define $B_i = \{b_j \in B \mid i \leq n - \ell_j\}$ and $D_i = \{d_{(i-1)3m+1}, \dots, d_{3im-\|B_i\|}\}$.
3. For each $k, 1 \leq k \leq m+1$, there is one voter of the form: $x \;\; c \;\; E_k \;\; F \;\; (E - E_k) \;\; G \;\; D \;\; B \;\; w$, where $E_k = \{e_{(3m-1)(k-1)+1}, \dots, e_{(3m-1)k}\}$ for each k, $1 \leq k \leq m+1$.
4. For each l, $1 \leq l \leq m-1$, there is one voter of the form: $F_l \;\; c \;\; (F - F_l) \;\; G \;\; D \;\; E \;\; B \;\; w \;\; x$, where $F_l = \{f_{(3m+1)(l-1)+1}, \dots, f_{(3m+1)l}\}$, for each l, $1 \leq l \leq m-1$.

In this election, candidate c is the unique level 2 BV winner with a level 2 score of $n + m + 1$.

We claim that $\mathcal{S}$ has an exact cover $\mathcal{S}'$ for B if and only if w can be made the unique BV winner of the resulting election by partition of voters (regardless of the tie-handling model used).

From left to right: Suppose $\mathcal{S}$ has an exact cover $\mathcal{S}'$ for B. Partition V the following way. Let V_1 consist of:

- the m voters of the first group that correspond to the exact cover (i.e., those m voters of the form $c \;\; S_i \;\; G_i \;\; (G - G_i) \;\; F \;\; D \;\; E \;\; (B - S_i) \;\; w \;\; x$ for which $S_i \in \mathcal{S}'$) and
- the $m+1$ voters of the third group (i.e., all voters of the form $x \;\; c \;\; E_k \;\; F \;\; (E - E_k) \;\; G \;\; D \;\; B \;\; w$.

Let $V_2 = V - V_1$. In subelection (C, V_1), candidate x is the unique level 1 BV winner. In subelection (C, V_2), candidate w is the first candidate who has a strict majority and moves on to the final round of the election. Thus there are w and x in the final run-off, which w wins with a strict majority on the first level. Since both subelections, (C, V_1) and (C, V_2), have unique BV winners, candidate w can be made the unique BV winner by partition of voters, regardless of the tie-handling model used.

From right to left: Suppose that w can be made the unique BV winner by exerting control by partition of voters (for concreteness, say in TP). Let (V_1, V_2) be such a successful partition. Since w wins the resulting two-stage election, w has to win at least one of the subelections (say, w wins (C, V_1)). If candidate c participates in the final round, he or she wins the election with a strict majority no later than on the second level, no matter which other candidates move forward to the final election. That means that in both subelections, (C, V_1) and (C, V_2), c must not be a BV winner. Only in the second voter group candidate w (who has to be a BV winner in (C, V_1)) gets points earlier than on the second-to-last level. So w has to be a level $3m+1$ BV winner in (C, V_1) via votes from the second voter group in V_1. As c scores already on the first two levels in voter groups 1 and 3, only x and the candidates in B can prevent c from winning in (C, V_2). However, since voters from the second voter group have to be in V_1 (as stated above), in subelection (C, V_2) only candidate x can prevent c from moving forward to the final round. Since x is always placed behind c in all votes except those votes from the third voter group, x has to be a level 1 BV winner in (C, V_2). In (C, V_1) candidate w gains all the points on exactly the $(3m+1)$st level, whereas the other candidates scoring more than one point up to this level receive their points on either earlier or later levels, so no candidate can tie with w on the $(3m+1)$st level and w

is the unique level $3m+1$ BV winner in (C, V_1). As both subelections, (C, V_1) and (C, V_2), have unique BV winners other than c, the construction works in model TE as well.

It remains to show that $\mathcal{S}$ has an exact cover $\mathcal{S}'$ for B. Since w has to win (C, V_1) with the votes from the second voter group, not all voters from the first voter group can be in V_1 (otherwise c would have n points already on the first level). On the other hand, there can be at most m voters from the first voter group in V_2 because otherwise x would not be a level 1 BV winner in (C, V_2). To ensure that no candidate in B has the same score as w, namely n points, and gets these points on an earlier level than w in (C, V_1), there have to be exactly m voters from the first group in V_2 and these voters correspond to an exact cover for B. ❑

Since Bucklin voting is a special case of fallback voting, we can answer one of the questions raised in [10] as follows:

Corollary 3.2. *Fallback voting is resistant to constructive control by partition of voters in model TP.*

The following construction will be used to handle the destructive case of control by partition of voters in model TP for fallback voting (see Theorem 3.5 below). The construction starts from an instance of Restricted Hitting Set, a restricted version of the NP-complete problem Hitting Set (see, e.g., [13]), which is defined as follows:

Name: Restricted Hitting Set (RHS).
Instance: A set $B = \{b_1, b_2, \ldots, b_m\}$, a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of nonempty subsets $S_i \subseteq B$ such that $n > m$, and a positive integer k with $1 < k < m$.
Question: Does $\mathcal{S}$ have a hitting set of size at most k, i.e., is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Note that by dropping the requirement "$n > m > k > 1$," we obtain the (unrestricted) Hitting Set problem. It is easy to see that Restricted Hitting Set is NP-complete.

Construction 3.3. *Let $(B, \mathcal{S}, k)$ be a given instance of* RHS*, with a set $B = \{b_1, b_2, \ldots, b_m\}$, a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of nonempty subsets $S_i \subseteq B$, and an integer k with $1 < k < m < n$. Define election (C, V), where $C = B \cup D \cup E \cup \{c, w\}$ is the candidate set with $D = \{d_1, \ldots, d_{2(m+1)}\}$ and $E = \{e_1, \ldots, e_{2(m-1)}\}$ and where V consists of the following $2n(k+1) + 4m + 2mk$ voters:*[1]

1. *For each i, $1 \leq i \leq n$, $k+1$ voters approve of $w \;\; S_i \;\; c$.*
2. *For each j, $1 \leq j \leq m$, one voter approves of $c \;\; b_j \;\; w$.*
3. *For each j, $1 \leq j \leq m$, $k-1$ voters approve of b_j.*
4. *For each p, $1 \leq p \leq m+1$, one voter approves of $d_{2(p-1)+1} \;\; d_{2p} \;\; w$.*
5. *For each r, $1 \leq r \leq 2(m-1)$, one voter approves of e_r.*
6. *$n(k+1) + m - k + 1$ voters approve of c.*
7. *$mk + k - 1$ voters approve of $c \;\; w$.*
8. *One voter approves of $w \;\; c$.*

[1]Recall: Disapproved candidates are omitted and approved candidates are ranked in the votes of a fallback election.

Note that $maj(V) = n(k+1) + 2m + mk + 1$. In election (C, V), only the two candidates c and w reach a strict majority, w on the third level and c on the second level (see Table 2). Thus c is the unique level 2 FV winner of election (C, V). Lemma 3.4 will be used in the proof of Theorem 3.5.

	c	$d_p \in D$	$e_r \in E$
$score^1$	$n(k+1) + 2m + mk$	≤ 1	1
$score^2$	$n(k+1) + 2m + mk + 1$	1	1
$score^{m+2}$	$2n(k+1) + 2m + mk + 1$	1	1
	w	$b_j \in B$	
$score^1$	$n(k+1) + 1$	$k-1$	
$score^2$	$n(k+1) + mk + k$	$\leq k + n(k+1)$	
$score^{m+2}$	$n(k+1) + 2m + mk + k + 1$	$\leq k + n(k+1)$	

Table 2: Level i scores in (C, V) for $i \in \{1, 2, m+2\}$.

Lemma 3.4. *In election (C, V) from Construction 3.3, for every partition of V into V_1 and V_2, candidate c is an FV winner of (C, V_1) or (C, V_2).*

Proof. For a contradiction, suppose that in both subelections, (C, V_1) and (C, V_2), candidate c is not an FV winner. Since $score^1_{(C,V)}(c) = \|V\|/2$, the two subelections satisfy that both $\|V_1\|$ and $\|V_2\|$ are even numbers, and that $score^1_{(C,V_1)}(c) = \|V_1\|/2$ and $score^1_{(C,V_2)}(c) = \|V_2\|/2$. Otherwise, c would have a strict majority already on the first level in one of the subelections and would win that subelection. For each $i \in \{1, 2\}$, c already on the first level has only one point less than the strict majority threshold $maj(V_i)$ in subelection (C, V_i), and c will get a strict majority in (C, V_i) no later than on the $(m+2)$nd level. Thus, for both $i = 1$ and $i = 2$, there must be candidates whose level $m+2$ scores in (C, V_i) are higher than the level $m+2$ score of c in (C, V_i). Table 2 shows the level $m+2$ scores of all candidates in (C, V). Only w and some $b_j \in B$ have a chance to beat c on that level in (C, V_i), $i \in \{1, 2\}$.

Suppose that c is defeated in both subelections by two distinct candidates from B (say, b_x defeats c in (C, V_1) and b_y defeats c in (C, V_2)). Thus the following must hold:[2]

$$\begin{aligned} score^{m+2}_{(C,V_1)}(b_x) + score^{m+2}_{(C,V_2)}(b_y) &\geq score^{m+2}_{(C,V)}(c) + 2 \\ 2n(k+1) + 2k - n(k+1) &\geq 2n(k+1) + mk + 2m + 3 \\ 2k &\geq n(k+1) + mk + 2m + 3, \end{aligned}$$

which contradicts our basic assumption $m > k > 1$. Thus the only possibility for c to not win any of the two subelections is that c is defeated in one subelection, say (C, V_1), by a candidate from B, say b_x, and in the other subelection, (C, V_2), by candidate w. Then it must hold that:[2]

$$score^{m+2}_{(C,V_1)}(b_x) + score^{m+2}_{(C,V_2)}(w) \geq score^{m+2}_{(C,V)}(c) + 2,$$

which is equivalent to

$$\begin{aligned} & 2n(k+1) + 2k + 2m + mk + 1 - n(k+1) - 1 \\ \geq \;\; & 2n(k+1) + mk + 2m + 3, \end{aligned}$$

i.e., $2k \geq n(k+1) + 3$. Since $n > 1$, this cannot hold, so c must be an FV winner in one of the subelections. ❑

[2]For the left-hand sides of the inequalities, note that each vote occurs in only one of the two subelections. To avoid double-counting those votes that give points to both candidates, we first sum up the overall number of points each candidate scores and then substract the double-counted points.

THEOREM 3.5. *Fallback voting is resistant to destructive control by partition of voters in model TP.*

PROOF. Susceptibility holds by [10, Lemma 3.4]. To prove NP-hardness in the TP case, we reduce RHS to our control problem. Consider the election (C, V) constructed according to Construction 3.3 from a given RHS instance $(B, \mathcal{S}, k)$, where $B = \{b_1, \dots, b_m\}$ is a set, $\mathcal{S} = \{S_1, \dots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$, and k is an integer with $1 < k < m < n$.

We claim that $\mathcal{S}$ has a hitting set $B' \subseteq B$ of size k if and only if c can be prevented from being the unique FV winner by partition of voters in model TP.

From left to right: Suppose $B' \subseteq B$ is a hitting set of size k for $\mathcal{S}$. Partition V into V_1 and V_2 as follows. Let V_1 consist of those voters of the second group where $b_j \in B'$ and of those voters of the third group where $b_j \in B'$. Let $V_2 = V - V_1$. In (C, V_1), no candidate reaches a strict majority (see Table 3), where $maj(V_1) = \lfloor k^2/2 \rfloor + 1$, and candidates c, w, and each $b_j \in B'$ win the election with an approval score of k.

	c	w	$b_j \in B'$	$b_j \notin B'$
score^1	k	0	$k-1$	0
score^2	k	0	k	0
score^3	k	k	k	0

Table 3: Level i scores in (C, V_1) for $i \in \{1, 2, 3\}$ and all candidates in $B \cup \{c, w\}$.

	c	$b_j \in B'$
score^1	$n(k+1) + 2m - k + mk$	0
score^2	$n(k+1) + 2m - k + mk + 1$	$\leq n(k+1)$
score^3	$\geq n(k+1) + 2m - k + mk + 1$	$\leq n(k+1)$
	w	$b_j \notin B'$
score^1	$n(k+1) + 1$	$k-1$
score^2	$n(k+1) + mk + k$	$\leq k + n(k+1)$
score^3	$n(k+1) + mk + 2m + 1$	$\leq k + n(k+1)$

Table 4: Level i scores in (C, V_2) for $i \in \{1, 2, 3\}$ and all candidates in $B \cup \{c, w\}$.

The level i scores in election (C, V_2) for $i \in \{1, 2, 3\}$ and all candidates in $B \cup \{c, w\}$ are shown in Table 4. Since in (C, V_2) no candidate from B wins, the candidates participating in the final round are $B' \cup \{c, w\}$. The scores in the final election $(B' \cup \{c, w\}, V)$ can be seen in Table 5. Since candidates c and w with the same level 2 scores are both level 2 FV winners, candidate c has been prevented from being the unique FV winner by partition of voters in model TP.

	c	w
score^1	$n(k+1) + 2m + mk$	$n(k+1) + m + 2$
score^2	$n(k+1) + 2m + mk + 1$	$n(k+1) + 2m + mk + 1$
	$b_j \in B'$	
score^1	$k-1$	
score^2	$\leq k + n(k+1)$	

Table 5: Level i scores in the final-stage election $(B' \cup \{c, w\}, V)$ for $i \in \{1, 2\}$.

From right to left: Suppose candidate c can be prevented from being a unique FV winner by partition of voters in model TP. From Lemma 3.4 it follows that candidate c participates in the final round. Since c has a strict majority of approvals, c has to be tied with or lose against another candidate by a strict majority at some level. Only candidate w has a strict majority of approvals, so w has to tie or beat c at some level in the final round. Because of the low scores of the candidates in D and E we may assume that only candidates from B are participating in the final round besides c and w. Let $B' \subseteq B$ be the set of candidates who also participate in the final round. Let ℓ be the number of sets in $\mathcal{S}$ not hit by B'. As w cannot reach a strict majority of approvals on the first level, we consider the level 2 scores of c and w: $score^2_{(B' \cup \{c,w\}, V)}(c) = n(k+1) + 2m + mk + 1 + \ell(k+1)$, and $score^2_{(B' \cup \{c,w\}, V)}(w) = n(k+1) + 2m + mk + k - \|B'\| + 1$. Since c has a strict majority already on the second level, w must tie or beat c on this level, so the following must hold:

$$\begin{aligned} score^2_{(B' \cup \{c,w\}, V)}(c) - score^2_{(B' \cup \{c,w\}, V)}(w) &\leq 0 \\ \|B'\| - k + \ell(k+1) &\leq 0. \end{aligned}$$

This is possible only if $\ell = 0$ (i.e., all sets in $\mathcal{S}$ are hit by B'), so $\|B'\| \leq k$. Thus $\mathcal{S}$ has a hitting set of size at most k. ❑

Finally, we turn to destructive control by partition of voters in model TE. The proof of Theorem 3.6 (which employs a reduction from DOMINATING SET) is omitted due to space.

THEOREM 3.6. *Bucklin voting (and thus fallback voting as well) is resistant to destructive control by partition of voters in model TE.*

4. CANDIDATE CONTROL IN BV

Theorem 4.1 strengthens the corresponding result for fallback voting [10].

THEOREM 4.1. *Bucklin voting is resistant to each of the 14 standard types of candidate control.*

For the hardness proofs showing Theorem 4.1, we again use the RHS problem defined in Section 3.

In this section, all reductions except one (namely that used to prove Lemma 4.2) will apply Construction 4.3 below. We first handle this one exception.

LEMMA 4.2. *Bucklin voting is resistant to constructive control by deleting candidates.*

PROOF. Susceptibility holds by Lemma 2.1. To prove NP-hardness of our control problem, we give a reduction from RHS. Let $(B, \mathcal{S}, k)$ be a RHS instance with a set $B = \{b_1, b_2, \dots, b_m\}$, a collection $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of nonempty subsets $S_i \subseteq B$, and a positive integer k satisfying $k < m < n$. Let $s_i = n + k - \|S_i\|$, $1 \leq i \leq n$, and $s = \sum_{i=1}^{n} s_i$. Note that all s_i are positive, since $m < n$.

Define election (C, V) with candidate set

$$C = B \cup C' \cup D \cup E \cup F \cup \{w\},$$

where $C' = \{c_1, c_2, \dots, c_{k+1}\}, D = \{d_1, d_2, \dots, d_s\}, E = \{e_1, e_2, \dots, e_n\}, F = \{f_1, \dots, f_{n+k}\}$, and let w be the distinguished candidate. Note that the number of candidates in D is $s = n^2 + kn - \sum_{i=1}^{n} \|S_i\|$. For each i, $1 \leq i \leq n$, let $D_i = \{d_{1+\sum_{j=1}^{i-1} s_j}, \dots, d_{\sum_{j=1}^{i} s_j}\}$, so $\|D_i\| = s_i$.

Define V to consist of the following $2(n+k+1)+1$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form: $S_i \;\; D_i \;\; w \;\; C' \;\; E \;\; (D - D_i) \;\; (B - S_i) \;\; F$.
2. For each j, $1 \leq j \leq k+1$, there is one voter of the form: $E \;\; (C' - \{c_j\}) \;\; c_j \;\; B \;\; D \;\; w \;\; F$.

3. There are $k+1$ voters of the form: $w \ F \ C' \ E \ B \ D$.

4. There are n voters of the form: $C' \ D \ F \ B \ w \ E$.

5. There is one voter of the form: $C' \ w \ D \ F \ E \ B$.

There is no unique BV winner in election (C, V), since w and the candidates in C' are level $n + k + 1$ BV winners.

We claim that $\mathcal{S}$ has a hitting set of size k if and only if w can be made the unique BV winner by deleting at most k candidates.

From left to right: Suppose $\mathcal{S}$ has a hitting set B' of size k. Delete the corresponding candidates. Now, w is the unique level $n + k$ BV winner of the resulting election.

From right to left: Suppose w can be made the unique BV winner by deleting at most k candidates. Since $k + 1$ candidates other than w have a strict majority on level $n + k + 1$ in election (C, V), after deleting at most k candidates, there is still at least one candidate other than w with a strict majority of approvals on level $n + k + 1$. However, since w was made the unique BV winner by deleting at most k candidates, w must be the unique BV winner on a level lower than or equal to $n + k$. This is possible only if in all n votes of the first voter group w moves forward by at least one position. This, however, is possible only if $\mathcal{S}$ has a hitting set B' of size k. ❑

Construction 4.3 will be applied to prove the remaining 13 cases of candidate control stated in Theorem 4.1.

CONSTRUCTION 4.3. *Let $(B, \mathcal{S}, k)$ be a given instance of* RHS*, where $B = \{b_1, b_2, \dots, b_m\}$ is a set, $\mathcal{S} = \{S_1, \dots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$ such that $n > m$, and $k < m$ is a positive integer. (Thus, $n > m > k > 1$.) Define election (C, V), where $C = B \cup \{c, d, w\}$ is the candidate set and where V consists of the following $6n(k + 1) + 4m + 11$ voters:*

1. *$2m + 1$ voters: $c\, d\, B\, w$.*
2. *$2n + 2k(n - 1) + 3$ voters: $c\, w\, d\, B$.*
3. *$2n(k + 1) + 5$ voters: $w\, c\, d\, B$.*
4. *For each i, $1 \le i \le n$, $2(k+1)$ voters: $d\, S_i\, c\, w\, (B - S_i)$.*
5. *For each j, $1 \le j \le m$, two voters: $d\, b_j\, w\, c\, (B - \{b_j\})$.*
6. *$2(k + 1)$ voters: $d\, w\, c\, B$.*

We now prove Theorem 4.1 (except for the case already handled separately in Lemma 4.2) via Construction 4.3, making use of the following lemma.

LEMMA 4.4. *Consider the election (C, V) constructed according to Construction 4.3 from a* RHS *instance $(B, \mathcal{S}, k)$.*

1. *c is the unique level 2 BV winner of $(\{c, d, w\}, V)$.*
2. *If $\mathcal{S}$ has a hitting set B' of size k, then w is the unique BV winner of election $(B' \cup \{c, d, w\}, V)$.*
3. *Let $D \subseteq B \cup \{d, w\}$. If c is not a unique BV winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that*
   (a) $D = B' \cup \{d, w\}$,
   (b) w is a level 2 BV winner of $(B' \cup \{c, d, w\}, V)$,
   (c) B' is a hitting set for $\mathcal{S}$ of size at most k.

PROOF. For the first part, note that there is no level 1 BV winner in election $(\{c, d, w\}, V)$ and we have the following level 2 scores in this election:

$$\begin{aligned} score^2_{(\{c,d,w\},V)}(c) &= 6n(k + 1) + 2(m - k) + 9, \\ score^2_{(\{c,d,w\},V)}(d) &= 2n(k + 1) + 4m + 2k + 3, \\ score^2_{(\{c,d,w\},V)}(w) &= 4n(k + 1) + 2m + 10. \end{aligned}$$

Since $n > m$ (which implies $n > k$), we have:

$$\begin{aligned} &score^2_{(\{c,d,w\},V)}(c) - score^2_{(\{c,d,w\},V)}(d) \\ &\quad = 4n(k + 1) - (2m + 4k) + 6 > 0, \\ &score^2_{(\{c,d,w\},V)}(c) - score^2_{(\{c,d,w\},V)}(w) \\ &\quad = 2n(k + 1) - (2k + 1) > 0. \end{aligned}$$

Thus, c is the unique level 2 BV winner of $(\{c, d, w\}, V)$.

For the second part, suppose that B' is a hitting set for $\mathcal{S}$ of size k. Then there is no level 1 BV winner in election $(B' \cup \{c, d, w\}, V)$, and we have the following level 2 scores:

$$\begin{aligned} score^2_{(B'\cup\{c,d,w\},V)}(c) &= 4n(k + 1) + 2(m - k) + 9, \\ score^2_{(B'\cup\{c,d,w\},V)}(d) &= 2n(k + 1) + 4m + 2k + 3, \\ score^2_{(B'\cup\{c,d,w\},V)}(w) &= 4n(k + 1) + 2(m - k) + 10, \\ score^2_{(B'\cup\{c,d,w\},V)}(b_j) &\le 2n(k + 1) + 2 \quad \text{for all } b_j \in B'. \end{aligned}$$

It follows that w is the unique level 2 BV winner of election $(B' \cup \{c, d, w\}, V)$.

For the third part, let $D \subseteq B \cup \{d, w\}$. Suppose c is not a unique BV winner of election $(D \cup \{c\}, V)$.

(3a) Other than c, only w has a strict majority of votes on the second level and only w can tie or beat c in $(D \cup \{c\}, V)$. Thus, since c is not a unique BV winner of election $(D \cup \{c\}, V)$, w is clearly in D. In $(D \cup \{c\}, V)$, candidate w has no level 1 strict majority, and candidate c has already on level 2 a strict majority. Thus, w must tie or beat c on level 2. For a contradiction, suppose $d \notin D$. Then

$$\begin{aligned} score^2_{(D\cup\{c\},V)}(c) &\ge 4n(k + 1) + 2m + 11; \\ score^2_{(D\cup\{c\},V)}(w) &= 4n(k + 1) + 2m + 10, \end{aligned}$$

which contradicts the observation that w ties or beats c on level 2. Thus, $D = B' \cup \{d, w\}$, where $B' \subseteq B$.

(3b) This part follows immediately from the proof of (3a).

(3c) Let ℓ be the number of sets in $\mathcal{S}$ not hit by B'. We have that $score^2_{(B'\cup\{c,d,w\},V)}(w) = 4n(k + 1) + 10 + 2(m - \|B'\|)$ and $score^2_{(B'\cup\{c,d,w\},V)}(c) = 2(m - k) + 4n(k+1)+9+2(k+1)\ell$. From part (3b) we know that

$$score^2_{(B'\cup\{c,d,w\},V)}(w) \ge score^2_{(B'\cup\{c,d,w\},V)}(c),$$

so $4n(k + 1) + 10 + 2(m - \|B'\|) \ge 2(m - k) + 4n(k + 1) + 9 + 2(k + 1)\ell$. This inequality implies $1 > \frac{1}{2} \ge \|B'\| - k + (k + 1)\ell$. Since $T = \|B'\| - k + (k + 1)\ell$ is an integer, we have $T \le 0$. If $T = 0$ then $\ell = 0$ and $\|B'\| = k$. Now assume $T < 0$. If $\ell = 0$, B' is a hitting set with $\|B'\| < k$, and if $\ell > 0$ then $(k + 1)\ell > k$, which contradicts $T = \|B'\| - k + (k + 1)\ell < 0$. In each possible case, we have a hitting set (as $\ell = 0$) of size at most k. ❑

Proof of Theorem 4.1. In each case, susceptibility holds by Lemma 2.1. For the four adding-candidates cases, NP-hardness follows immediately from Lemma 4.4.

NP-hardness for constructive control by deleting candidates has been shown in Lemma 4.2. To show the problem NP-hard in the destructive case, let (C,V) be the election resulting from a RHS instance $(B,\mathcal{S},k)$ according to Construction 4.3, and let c be the distinguished candidate. We claim that $\mathcal{S}$ has a hitting set of size at most k if and only if c can be prevented from being a unique BV winner by deleting at most $m-k$ candidates.

From left to right: Suppose $\mathcal{S}$ has a hitting set B' of size k. Delete the $m-k$ candidates $B-B'$. Now, both candidates c and w have a strict majority on level 2, but

$$\begin{aligned} score^2_{(\{c,d,w\}\cup B',V)}(c) &= 4n(k+1)+2(m-k)+9, \\ score^2_{(\{c,d,w\}\cup B',V)}(w) &= 4n(k+1)+2(m-k)+10, \end{aligned}$$

so w is the unique level 2 BV winner of this election.

From right to left: Suppose that c can be prevented from being a unique BV winner by deleting at most $m-k$ candidates. Let $D' \subseteq B \cup \{d,w\}$ be the set of deleted candidates (so $c \notin D'$) and $D = (C-D') - \{c\}$. It follows immediately from Lemma 4.4 that $D = B' \cup \{d,w\}$, where B' is a hitting set for $\mathcal{S}$ of size at most k.

To show that Bucklin voting is resistant to constructive (or destructive) control by partition/run-off partition of candidates in TE and TP, map the instance $(B,\mathcal{S},k)$ to the instance $((C,V),w)$ (or $((C,V),c)$), where (C,V) is the election from Construction 4.3. NP-hardness now follows from Lemma 4.4; the detailed argument is omitted due to space limitations (note that, in particular, if $\mathcal{S}$ has a hitting set of size k, partitioning $C=(C_1,C_2)$ into $C_1 = B' \cup \{c,d,w\}$ and $C_2 = C - C_1$ will be successful). ❑ Theorem 4.1

5. ADDING/DELETING VOTERS IN BV

Finally, we turn to control by adding voters and by deleting voters for Bucklin voting. As with fallback voting [10], we have resistance in the constructive cases and vulnerability in the destructive cases. Since Bucklin voting is a special case of fallback voting, the two resistance results in Theorem 5.1 (which both are shown via a reduction from X3C) strengthen the corresponding results for fallback voting [10] and the two vulnerability results immediately follow from the corresponding results for fallback voting [10]. The proof of Theorem 5.1 is omitted due to space limitations.

THEOREM 5.1. *Bucklin voting is resistant to constructive control by adding voters and by deleting voters and is vulnerable to destructive control by adding voters and by deleting voters.*

6. CONCLUSIONS

Solving the three open questions of Erdélyi and Rothe [10], we have shown that fallback voting is fully resistant to control by partition of voters. Thus, among natural voting systems with a polynomial-time winner problem, fallback voting has the most proven resistances to control. SP-AV is known to have an almost as broad control resistance [9]; however, fallback voting is arguably more natural than SP-AV. We have also studied the control complexity of Bucklin voting, thus improving the corresponding resistance results for fallback voting. One case of control by partition of voters (namely, the destructive case in model TP) remains open for Bucklin voting. It would also be interesting and challenging to complement our worst-case hardness results by theoretical and empirical typical-case studies of these problems.

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