Monotone maps, sphericity and bounded second eigenvalue

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Abstract

We consider *monotone* embeddings of a finite metric space into low-dimensional normed space. That is, embeddings that respect the order among the distances in the original space. Our main interest is in embeddings into Euclidean spaces. We observe that any metric on n points can be embedded into l_2^n, while (in a sense to be made precise later), for almost every n-point metric space, every monotone map must be into a space of dimension $\Omega(n)$ (Lemma 3).

It becomes natural, then, to seek explicit constructions of metric spaces that cannot be monotonically embedded into spaces of sublinear dimension. To this end, we employ known results on *sphericity* of graphs, which suggest one example of such a metric space—that is defined by a complete bipartite graph. We prove that an δn-regular graph of order n, with bounded diameter has sphericity $\Omega(n/(\lambda_2 + 1))$, where λ_2 is the second largest eigenvalue of the adjacency matrix of the graph, and $0 < \delta \leqslant \frac{1}{2}$ is constant (Theorem 4). We also show that while random graphs have linear sphericity, there are *quasi-random* graphs of logarithmic sphericity (Lemma 7).

For the above bound to be linear, λ_2 must be constant. We show that if the second eigenvalue of an $n/2$-regular graph is bounded by a constant, then the graph is close to being complete bipartite. Namely, its adjacency matrix differs from that of a complete bipartite graph in only $o(n^2)$ entries

(Theorem 5). Furthermore, for any $0 < \delta < \frac{1}{2}$, and λ_2, there are only finitely many δn-regular graphs with second eigenvalue at most λ_2 (Corollary 4).

1. Introduction

Euclidean embeddings of finite metric spaces have been extensively studied, with the aim of finding an embedding that does not distort the metric too much. We refer the reader to the survey papers of Indyk [11] and Linial [13], as well as Chapter 15 of Matoušek's Discrete Geometry book [16]. Here we focus on different types of embeddings. Namely, those that preserve the order relation of the distances. We call such embeddings *monotone*. There are quite a few applications that make this concept natural and interesting, since there are numerous algorithmic problems whose solution depends only on the order among the distances. Specifically, questions that concern nearest neighbors. The notion of monotone embeddings suggests the following general strategy toward the resolution of such problems. Namely, embed the metric space at hand monotonically into a "nice" space, for which good algorithms are known to solve the problem. Solve the problem in the "nice" space—the same solution applies as well for the original space. "Nice" often means a low-dimensional normed space. Thus, we focus on the minimal dimension which permits a monotone embedding.

In Section 2 we observe that any metric on n points can be monotonically embedded into an n-dimensional Euclidean space, and that the bound on the dimension is asymptotically tight. The embedding clearly depends only on the order of the distances (Lemma 1). We show that for almost every ordering of the $\binom{n}{2}$ distances among n points, the host space of a monotone embedding must be $\Omega(n)$-dimensional. Similar bounds are given for embeddings into l_∞, and some bounds are also deduced for other norms.

Next we consider embeddings that are even less constrained. Given a metric space (X, δ) and some threshold t, we seek a mapping f that only respects this threshold. Namely, $\|f(x) - f(y)\| < 1$ iff $\delta(x, y) < t$. The input to this problem can thus be thought of as a graph (adjacency indicating distances below the threshold t). The minimal dimension d, such that a graph G can be mapped this way into l_2^d is known as the *sphericity* of G, and denoted $Sph(G)$. Reiterman et al. [20] and Maehara [15] show that the sphericity of $K_{n,n}$ is at least n. This is, then, an explicit example of a metric space which requires linear dimension to be monotonically embedded into l_2. Other than that, the best lower bounds previously known to us are logarithmic. In Section 3 we prove a novel lower bound, namely that for $0 < \delta \leqslant \frac{1}{2}$, $Sph(G) = \Omega(\frac{n}{\lambda_2+1})$, for any n-vertex δn-regular graph, with bounded diameter. Here λ_2 is the second largest eigenvalue of the graph. We also show examples of quasi-random graphs of logarithmic sphericity. This is somewhat surprising since quasi-random graphs tend to behave like random graphs, yet the latter have linear sphericity.

In our search for further examples of graphs of linear sphericity, we investigate in Section 4 families of graphs whose second eigenvalue is bounded by a constant (for which the aforementioned lower bound is linear). We show that such graphs are close to being complete

bipartite, in the sense that one needs to modify only $o(n^2)$ entries in the adjacency matrix to get the latter from the former. As a corollary, we get that for $0 < \delta < \frac{1}{2}$, and λ_2, there are only finitely many δn-regular graphs with second eigenvalue at most λ_2.

2. Monotone maps

2.1. Definitions

Let $X = ([n], \delta)$ be a metric space on n points, such that all pairwise distances are distinct. Let $\| \cdot \|$ be a norm on $\mathbb{R}^d$. We say that $\phi : X \to (\mathbb{R}^d, \| \cdot \|)$ is a *monotone map* if for every $w, x, y, z \in X$, $\delta(x, y) < \delta(w, z) \Leftrightarrow \|\phi(x) - \phi(y)\| < \|\phi(w) - \phi(z)\|$.

We denote by $d(X, \| \cdot \|)$ the minimal t such that there exists a monotone map from X to $(\mathbb{R}^t, \| \cdot \|)$. We denote by $d(n, \| \cdot \|) = \max_X d(X, \| \cdot \|)$, the smallest dimension to which every n point metric can be mapped monotonically.

The first thing to note is that we are actually concerned only with the *order* among the distances between the points in the metric space, and not with the actual distances. Let (X, δ) be a finite metric space, and let ρ be a linear order on $\binom{X}{2}$. We say that ρ and (X, δ) are *consistent* if for every $w, x, y, z \in X$, $\delta(x, y) < \delta(w, z) \Leftrightarrow (x, y) <_\rho (w, z)$.

We start with an easy, but useful observation.

Lemma 1. *Let X be a finite set. For every linear order relation ρ on $\binom{X}{2}$, there exists a distance function δ on X, that is consistent with ρ.*

Proof. Let $\{\varepsilon_{ij}\}_{(i,j) \in \binom{X}{2}}$ be small, non-negative numbers, ordered as per ρ. Define $\delta(i, j) = 1 + \varepsilon_{ij}$. It is obvious that δ induces the desired order on the distances of X, and, that if the ε's are small, the triangle inequality holds. $\square$

When we later (Section 2.3) use this observation, we refer to it as a *standard ε-construction*, where $\varepsilon = \max \varepsilon_{ij}$. It is not hard to see that this metric is Euclidean, that is, the resulting metric can be isometrically embedded into l_2, see Lemma 3 below.

We say that an order relation ρ on $\binom{[n]}{2}$ is *realizable* in $(\mathbb{R}^d, \| \cdot \|)$ if there exists a metric space (X, δ) on n points which is consistent with ρ, and a monotone map $\phi : X \to \mathbb{R}^d$. We say that ϕ is a realization of ρ. (Thus, $d(n, \| \cdot \|)$ is the minimal d such that any linear order on $\binom{[n]}{2}$ is realizable in $(\mathbb{R}^d, \| \cdot \|)$.)

We denote by $J = J_n$ the $n \times n$ all ones matrix, and by PSD_n the cone of real symmetric $n \times n$ positive semidefinite matrices. We omit the subscript n when it is clear from the context.

Finally, for a graph G, and U, V subsets of its vertices, we denote by $e(U, V) = |\{(u, v) \in E(G) : u \in U, v \in V\}|$, and $e(U) = |\{(u, u') \in E(G) : u, u' \in U\}|$.

2.2. Monotone maps into l_∞

Lemma 2. *The minimal dimension required to monotonically embed n points into l_∞ is bounded by: $\frac{n}{2} - 1 \leqslant d(n, l_\infty) \leqslant n$.*

Proof. It is well known that any metric X on n points can be embedded into l_∞^n isometrically, hence $d(n, l_\infty) \leqslant n$.

For the lower bound, we define a metric space (X, δ) with $2n + 2$ points that cannot be realized in l_∞^n. By Lemma 1, it suffices to define an ordering on the distances. In fact, we define only a partial order, any linear extension of which will do. The $2n + 2$ points come in $n + 1$ pairs, $\{x_i, y_i\}_{i=1,\ldots,n+1}$. If $z \notin \{x_i, y_i\}$, we let $\delta(x_i, y_i) > \delta(x_i, z), \delta(y_i, z)$. Assume for contradiction that a monotone map ϕ into l_∞^n does exist. For each pair (x, y) define $j(x, y)$ to be some index i for which $|\phi(x)_i - \phi(y)_i|$ is maximized, that is, an index i for which $|\phi(x)_i - \phi(y)_i| = \|\phi(x) - \phi(y)\|_\infty$.

By the pigeonhole principle there exist two pairs, say (x_1, y_1) and (x_2, y_2), for which $j(x_1, y_1) = j(x_2, y_2) = j$. It is easy to verify that our assumptions on the four real numbers $\phi(x_1)_j, \phi(x_2)_j, \phi(y_1)_j, \phi(y_2)_j$, are contradictory. Thus $d(n, l_\infty) \geqslant \frac{n}{2} - 1$.   □

2.3. Monotone maps into l_2

Lemma 3. *The minimal dimension required to monotonically embed n points into l_2 is bounded by:* $\frac{n}{2} \leqslant d(n, l_2) \leqslant n$. *Furthermore, for every $\delta > 0$, and every large enough n, almost no linear orders ρ on $\binom{[n]}{2}$ can be realized in dimension less than $\frac{n}{2+\delta}$.*

Note 1. *The upper bound is apparently folklore. As we could not find a reference for it, we give a proof here.*

The second part of the lemma relies on a bound on the number of *sign-patterns* of a sequence of real polynomials. Let $p_1, \ldots, p_m$ be real polynomials in l variables of (total) degree d, and let $x \in \mathbb{R}^l$ be a point where none of them vanish. The sign-pattern at x is $(\operatorname{sgn}(p_1(x)), \ldots, \operatorname{sgn}(p_m(x)))$. Denote the total number of different sign-patterns that can be obtained from $p_1, \ldots, p_m$ by $s(p_1, \ldots, p_m)$. A variation of the Milnor–Thom theorem [17] due to Alon et al. [1]:

Theorem 1 (*Alon et al. [1]*). *Let $p_1, \ldots, p_m$ be real polynomials as above. Then for any integer k between 1 and m:*

$$s(p_1, \ldots, p_m) \leqslant 2kd \cdot (4kd - 1)^{l + \frac{m}{k} - 1}.$$

Proof. Let ρ be a linear order on $\binom{[n]}{2}$. Let ε be a real symmetric matrix with the following properties:

- $\varepsilon_{ii} = 0$ for all i.
- $\frac{1}{n} > \varepsilon_{ij} > 0$, for all $i \neq j$.
- The numbers $\varepsilon_{i,j}$ are consistent with the order ρ.

Since the sum of each row is strictly less than one, all eigenvalues of ε are in the open interval $(-1, 1)$. It follows that the matrix $I - \varepsilon$ is positive definite. Therefore, there exists a matrix V such that $VV^t = I - \varepsilon$. Denote the ith row of V by v_i. Clearly, the v_i's are unit vectors, and $\langle v_i, v_j \rangle = -\varepsilon_{i,j}$ for $i \neq j$. Therefore, $\|v_i - v_j\|_2^2 = \langle v_i, v_i \rangle + \langle v_j, v_j \rangle - 2\langle v_i, v_j \rangle = 2 + 2\varepsilon_{i,j}$. It follows that the map $\phi(i) = v_i$ is a realization of ρ, and the upper bound is

proved. In fact, one can add another point without increasing the dimension, by mapping it to 0, and perturbing the diagonal.

For the lower bound, it is implicit in [15,20] (see Theorem 2 below) that if X is the metric induced by $K_{n,n}$, then $d(X, l_2) \geqslant n$.

For the second part of the lemma, set $n = c \cdot d$, for some constant c, and $l = n \cdot d$. Consider a point $x \in R^l$, and think of it as an $n \times d$ matrix. Denote the ith row of this matrix by x_i. As before, x *realizes* an order ρ on $\binom{[n]}{2}$ if the distances $\|x_i - x_j\|$ are consistent with ρ.

For two different pairs, (i_1, j_1) and (i_2, j_2), define the polynomial

$$p_{(i_1, j_1), (i_2, j_2)}(x) = \|x_{i_1} - x_{j_1}\|^2 - \|x_{i_2} - x_{j_2}\|^2.$$

The list contains $m = \binom{n}{2}$ polynomials of degree 2. Note that there is a 1:1 correspondence between orders on $\binom{[n]}{2}$ and sign-patterns of $p_1, \ldots, p_m$, thus no more than $s = s(p_1, \ldots, p_m)$ orders may be realized in l_2^d.

Take $k = \mu n^2$, for some large constant μ. By Theorem 1 $\log s$ is approximately $2cd^2 \log d$. By contrast, that total number of orders is $\binom{n}{2}!$, so its log is about $c^2 d^2 \log d$. If c is bigger than 2, almost all order relations cannot be realized. $\square$

Note 2. *In fact, the same proof shows that for any positive integer t, almost all orders on $\binom{n}{2}$ require linear dimension to be realized, and in particular that $d(n, l_{2t}) = \Omega(n)$ (where the constant of proportionality depends only on t): Simply repeat the argument above with polynomials of degree 2t rather than quadratic polynomials.*

2.4. Other norms

We conclude this section with two easy observations about monotone maps into other normed spaces. The first gives an upper bound on the dimension required for embedding into l_p:

Lemma 4. *The minimal dimension required to monotonically embed n points into l_p is bounded by: $d(n, l_p) \leqslant \binom{n}{2}$.*

Proof. By Lemma 3, any metric space on n points can be mapped monotonically into l_2. It is known (see [5] and also Chapter 15 of [16]) that any l_2 metric on n points can be isometrically embedded into $\binom{n}{2}$-dimensional l_p. The composition of these mappings is a monotone mapping of the metric space into $\binom{n}{2}$-dimensional l_p. $\square$

The second observation gives a lower bound for arbitrary norms. We first note the following:

Lemma 5. *Let $\|\cdot\|$ be an arbitrary n-dimensional norm and let $x_1, \ldots, x_{5^n}$ be points in $\mathbb{R}^n$, such that $\|x_i - x_j\| > 1$ for all $i \neq j$. Then there exits a pair (x_i, x_j) such that $\|x_i - x_j\| \geqslant 2$.*

Proof. Denote by v the volume of B, the unit ball in $(\mathbb{R}^n, \|\cdot\|)$. The translates $x_i + \frac{1}{2}B$ are obviously non-intersecting, so the volume of their union is $(\frac{5}{2})^n v$. Assume for contradiction that all pairwise distances are less than 2, then all these balls are contained in a single ball of radius less than $\frac{5}{2}$. But this is impossible, since the volume of this ball is less than $(\frac{5}{2})^n v$. $\square$

Note that the l_∞ norm shows that indeed an exponential number of points is required for the lemma to follow. We do not know, however, the smallest base of the exponent for which the claim holds. The determination of this number seems to be of some interest.

Corollary 1. *There exists an n-point metric space* (X, δ) *such that for any norm* $\|\cdot\|$, $d(n, \|\cdot\|) = \Omega(\log n)$.

Proof. We construct a distance function on $5^n + 1$ points which cannot be realized in any n-dimensional norm. By Lemma 1 it suffices to define a partial order on the distances. Denote the points in the metric space $0, \ldots, 5^n$. Let the distance between 0 and any other point be smaller than any distance between any two points $i \neq j > 0$. Consider a monotone map ϕ of the metric space into n-dimensional normed space. Assume, w.l.o.g., that $\min_{1 \leqslant i < j \leqslant 5^n} \|\phi(i) - \phi(j)\| = 1$. By the previous lemma there exists a pair of points, $i, j \neq 0$, such that $\|\phi(i) - \phi(j)\| > 2$. But for ϕ to be monotone it must satisfy $\|\phi(0) - \phi(i)\| < 1$ and $\|\phi(0) - \phi(j)\| < 1$, contradicting the triangle inequality. $\square$

3. Sphericity

So far we have concentrated on embeddings of a metric space into a normed space, that preserve the order relations between distances. However, in the examples that gave us the lower bounds for l_∞ and for arbitrary norms, we actually only needed to distinguish between "long" and "short" distances. This motivates the introduction of a broader class of maps, that need only respect the distinction between short and long distances. More formally, let $X = ([n], \delta)$ be a metric space. Its *proximity graph* with respect to some threshold τ, is a graph on n vertices, with an edge between i and j iff $\delta(i, j) \leqslant \tau$. An embedding of a proximity graph, is a mapping ϕ of its vertices into normed space, such that $\|\phi(i) - \phi(j)\| < 1$ iff (i, j) is an edge in the proximity graph (we assume that no distance is exactly 1). The minimal dimension in which a graph can be so embedded (in Euclidean space) was first studied by Maehara [14] under the name *sphericity*, and denoted $Sph(G)$. Following this terminology, we call such an embedding *spherical*.

The sphericity of graphs was further studied by Maehara and Frankl [7], Maehara [15], and Reiterman et al. [19–21]. Breu and Kirkpatrick have shown in [3] that it is NP-hard to recognize graphs of sphericity 2 (also known as *unit disk graphs*) and graphs of sphericity 3. We refer the reader to [19] for a survey of results regarding this parameter, and mention only a few of them here.

Theorem 2. *Let G be graph on n vertices with minimal degree* δ. *Let* λ_n *be the least eigenvalue of its adjacency matrix.*
1. $Sph(K_{m,n}) \leqslant m + \frac{n}{2} - 1$ [14].

2. $Sph(G) = O(\lambda_n^2 \log n)$ [7].
3. $Sph(G) = O((n - \delta) \log(n - \delta))$ [19].
4. $Sph(K_{n,n}) \geqslant n$ [15,20].
5. *All but a $\frac{1}{n}$ fraction of graphs on $n > 37$ vertices have sphericity at least $\frac{n}{15} - 1$* [19].
6. $Sph(G) \geqslant \frac{\log \alpha(G)}{\log(2r(G)+1)}$, *where $\alpha(G)$ is the independence number of G, and $r(G)$ is its radius* [20].

The first thing to note is that any lower bower on the sphericity of some graph on n vertices is also a lower bound on $d(n, l_2)$. In particular, the fact that $Sph(K_{n,n}) \geqslant n$ proves the lower bound in Lemma 3. (Similarly, any upper bound on the former also applies to the latter.)

In this section we are interested in graphs of large sphericity. The above results tell us that they exist in abundance, yet that graphs of very small maximal degree or very large minimal degree have small sphericity (the maximal degree is an upper bound on $|\lambda_n|$, hence by (2) the sphericity is small if all degrees are small). Other than the complete bipartite graph, the above results do not point out an explicit graph with super-logarithmic sphericity.

3.1. Upper bound on margin

Following Frankl and Maehara [7], consider an embedding of a proximity graph where there is a large margin between short and long distances. In such a situation, the Johnson–Lindenstrauss Lemma [12] would yield a spherical embedding into lower dimension: It allows reducing the dimension at the cost of some distortion. If the distortion is small with respect to the margin, the short and long distances remain separated. Alas, we show that for most regular graphs this margin is not large enough for the method to be useful:

Theorem 3. *Let G be a δn-regular graph, with second eigenvalue $\lambda_2 > \frac{2}{n}$. Let ϕ be an embedding of G as a proximity graph. Denote $a = \max_{u \sim v} \|\phi(u) - \phi(v)\|_2^2$, and $b = \min_{u \nsim v} \| \phi(u) - \phi(v)\|_2^2$. Then $b - a = O(\frac{\lambda_2 + \delta}{\delta n})$.*

Proof. Denote $m = \min\{1 - a, b - 1\}$, and for a vertex i, denote $v_i = \phi(i)$. The largest value m can attain, over all embeddings ϕ, is given by the following quadratic semidefinite program [3] (and is attained when $1 - a = 1 - b$):

$$
\begin{aligned}
\max \ & m \\
\text{s.t.} \ & \forall (i, j) \in E(G), \quad \|v_i - v_j\|^2 \leqslant 1 - m, \\
& \forall (i, j) \notin E(G), \quad \|v_i - v_j\|^2 \geqslant 1 + m.
\end{aligned}
$$

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[3] For reference on semidefinite programming see [8].

Its dual turns out to be

$$
\begin{aligned}
\min \ & \tfrac{1}{2} tr\, A \\
\text{s.t.} \ & A \in PSD, \\
& \forall\, (i, j) \in E(G), \quad A_{ij} \leqslant 0, \\
& \forall\, (i, j) \notin E(G), \ i \neq j, \quad A_{ij} \geqslant 0, \\
& \forall\, i, \quad \sum_{j=1,\dots,n} A_{ij} = 0, \\
& \sum_{i \neq j} |A_{ij}| = 1.
\end{aligned}
$$

Equivalently, we can drop the last constraint, and change the objective function to $\min \frac{tr\, A}{\sum_{i \neq j} |A_{ij}|}$. Next we construct an explicit feasible solution for the dual program, and conclude from it a bound on m.

Let M be the adjacency matrix of G. Define $A = I + \alpha J - \beta M$. To satisfy the constraints we need

$$
\begin{aligned}
& A \in PSD, \\
& \beta \geqslant \alpha \geqslant 0, \\
& 1 + \alpha n - \beta \delta n = 0.
\end{aligned}
$$

The last condition implies $\alpha = \beta \delta - \frac{1}{n}$, so it follows that $\beta \geqslant \alpha$, and the constraint on β is $\beta \geqslant \frac{1}{\delta n}$.

Now, since we assume that the graph is δn-regular, its Perron eigenvector is $\vec{1}$, corresponding to eigenvalue δn. Therefore, we can consider the eigenvectors of M to be eigenvectors of J and I as well, and hence also eigenvectors of A. If $\lambda \neq \delta n$ is an eigenvalue of M, then $1 - \beta \lambda$ is an eigenvalue of A, corresponding to the same eigenvector. Denote by λ_2 the second largest eigenvalue of M, then in order to satisfy the condition $A \in PSD$ it is enough to set $\beta = \frac{1}{\lambda_2}$, in which case all the constraints are fulfilled.

We conclude that

$$
\begin{aligned}
m \leqslant \frac{tr\, A}{\sum_{i \neq j} |A_{ij}|} &= \frac{n(1 + \alpha)}{\delta n^2 (\beta - \alpha) + ((1 - \delta)n^2 - n)\alpha} \\
&= \frac{n + \frac{\delta n}{\lambda_2} - 1}{\delta n (\frac{n + \delta n}{\lambda_2} - 1) + ((1 - \delta)n - 1)(\frac{\delta n}{\lambda_2} - 1)} < 4\frac{1 + \frac{\delta}{\lambda_2}}{\frac{\delta n}{\lambda_2}} = 4\frac{\lambda_2 + \delta}{\delta n}.
\end{aligned}
$$

In particular, $b - a = O(\frac{\lambda_2 + \delta}{\delta n})$. $\quad\square$

In order to derive a non-trivial result from the Johnson–Lindenstrauss Lemma, we need that $\frac{1}{m^2} \log n = o(n)$, and in particular that $m = \omega(\sqrt{\log n / n})$. The above shows that this can happen only if $\lambda_2 = \omega(\delta\sqrt{n \log n})$. On the other hand, Frankl and Maehara show that their method does give a non-trivial bound when $\lambda_n = o(\sqrt{\frac{n}{\log n}})$. Consequently, we get that a δn-regular graph (think of δ as constant) cannot have both $\lambda_2 = o(\sqrt{n \log n})$ and $\lambda_n = o(\sqrt{\frac{n}{\log n}})$. This is a bit more subtle than what one gets from the second moment argument, namely, that the graph cannot have both $\lambda_2 = o(\sqrt{n})$ and $\lambda_n = o(\sqrt{n})$.

3.2. Lower bound on sphericity

Theorem 4. *Let G be a d-regular graph with diameter D and λ_2, the second largest eigenvalue of G's adjacency matrix, at least $d - \frac{1}{2}n$. Then $Sph(G) = \Omega(\frac{d-\lambda_2}{D^2(\lambda_2+O(1))})$.*

In the interesting range where $d \leqslant \frac{n}{2}$, and $\lambda_2 \geqslant 1$ the bound is $Sph(G) = \Omega(\frac{d-\lambda_2}{D^2\lambda_2})$.

In proving the theorem will need the following lemma (see [10, p. 175]):

Lemma 6. *Let X be a real symmetric matrix, then $rank(X) \geqslant \frac{(tr X)^2}{\sum_{i,j} X_{i,j}^2}$.*

Proof. It will be useful to consider the following operation on matrices. Let A be an $n \times n$ symmetric matrix, and denote by $\vec{a}$ the vector whose ith coordinate is A_{ii}. Define $R(A)$ to be the $n \times n$ matrix with all rows equal to $\vec{a}$, and $C(A) = R(A)^t$. Define

$$\check{A} = 2A - C(A) - R(A) + J.$$

First note that the rank of $\check{A}$ and that of A can differ by at most 3. Now, consider the case where A is the Gram matrix of some vectors $v_1, \ldots, v_n \in R^d$. Then all diagonal entries of $\check{A}$ equal one, and the (i, j) entry is $2\langle v_i, v_j \rangle - \langle v_i, v_i \rangle - \langle v_j, v_j \rangle + 1 = 1 - \|v_i - v_j\|^2$.

Applying Lemma 6 to $\check{A}$, we conclude that

$$rank(\check{A}) \geqslant \frac{n^2}{n + \sum_{i \neq j}(1 - \|v_i - v_j\|^2)^2}. \tag{1}$$

Let $v_1, \ldots, v_n \in \mathbb{R}^d$ be an embedding of G. By the discussion above it is enough to show that

$$\sum_{i \neq j}(1 - \|v_i - v_j\|^2)^2 = O\left(D^2 n^2 \frac{\lambda_2}{d - \lambda_2}\right). \tag{2}$$

By the triangle inequality $\|v_i - v_j\| \leqslant D$ for any two vertices. So the LHS of (2) is bigger by at most a factor of D^2 than

$$\sum_{(i,j) \notin E}(\|v_i - v_j\|^2 - 1) + \sum_{(i,j) \in E}(1 - \|v_i - v_j\|^2)$$

$$= \sum_{(i,j) \notin E}\|v_i - v_j\|^2 - \sum_{(i,j) \in E}\|v_i - v_j\|^2 - \binom{n}{2} + nd. \tag{3}$$

We can bound this sum from above, by solving the following SDP:

$$\max \sum_{(i,j) \notin E}(V_{ii} + V_{jj} - 2V_{ij}) + \sum_{(i,j) \in E}(-V_{ii} - V_{jj} + 2V_{ij}) - \binom{n}{2} + nd$$

$$\text{s.t. } V \in PSD,$$

$$\forall (i, j) \in E, \quad V_{ii} + V_{jj} - 2V_{ij} \leqslant 1,$$

$$\forall (i, j) \notin E, \quad V_{ii} + V_{jj} - 2V_{ij} \geqslant 1.$$

The dual problem is

$$\min \tfrac{1}{2} tr\, A$$
$$\text{s.t. } A \in PSD,$$
$$\forall\, (i, j) \in E, \quad A_{ij} \leqslant -1,$$
$$\forall\, (i, j) \notin E, i \neq j, \quad A_{ij} \geqslant 1,$$
$$\forall\, i \in [n], \quad \sum_{j=1,\dots,n} A_{ij} = 0.$$

Let M by the adjacency matrix of the graph, and set $A = (\alpha d - n)I + J - \alpha M$, where $\alpha \geqslant 2$ will be determined shortly. This takes care of the all constraints except for $A \in PSD$. Note that since M is regular, its eigenvectors are also eigenvectors of A. Moreover, if $Mu = \lambda u$ for a non-constant u, then $Au = (\alpha d - n - \alpha \lambda)u$ (and $A\vec{1} = \vec{0}$). So take $\alpha = \frac{n}{d - \lambda_2}$, and by our assumption on λ_2, $\alpha \geqslant 2$.

Now A gives an upper bound on (3):

$$\frac{1}{2} tr\, A = \frac{1}{2} n(\alpha d - n + 1) = \frac{1}{2} n^2 \frac{d}{d - \lambda_2} - \frac{1}{2} n^2 + \frac{1}{2} n = \frac{1}{2} n^2 \frac{\lambda_2}{d - \lambda_2} + \frac{1}{2} n.$$

This, by (1), shows that the dimension of the embedding is $\Omega\left(\frac{d - \lambda_2}{D^2(\lambda_2 + O(1))}\right)$. $\square$

3.3. A quasi-random graph of logarithmic sphericity

It is an intriguing problem to construct new examples of graphs of linear sphericity. Since random graphs have this property, it is natural to search among quasi-random graphs. There are several equivalent definitions for such graphs (see [2]). The one we adopt here is:

Definition 3.1. A family of graphs is called *quasi-random* if the graphs in the family are $(1 + o(1))\frac{n}{2}$-regular, and all their eigenvalues except the largest one are (in absolute value) $o(n)$.

Counter-intuitively, perhaps, quasi-random graphs may have very small sphericity.

Lemma 7. *Let $\mathbb{G}$ be the family of graphs with vertex set $\{0, 1\}^k$, and edges connecting vertices that are at Hamming distance at most $\frac{k}{2}$. Then $\mathbb{G}$ is a family of quasi-random graphs of logarithmic sphericity.*

Proof. The fact that the sphericity is logarithmic is obvious—simply map each vertex to the vector in $\{0, 1\}^k$ associated with it. To show that all eigenvalues except the largest one are $o(2^k)$ we need the following facts about Krawtchouk polynomials (see [23]). Denote by $K_s^{(k)}(i) = \sum_{j=0}^{s}(-1)^j \binom{i}{j}\binom{k-i}{s-j}$ the Krawtchouk polynomial of order s over $\mathbb{Z}_2^k$. For simplicity we assume that k is odd.

1. For any $x \in \mathbb{Z}_2^k$ with $|x| = i$, $\sum_{z \in \mathbb{Z}_2^k, |z| = s}(-1)^{\langle x, z \rangle} = K_s^{(k)}(i)$.
2. $\sum_{s=0}^{l} K_s^{(k)}(i) = K_l^{(k-1)}(i - 1)$.
3. For any s and k, $\max_{i=0,\dots,n} |K_s^{(k)}(i)| = K_s^{(k)}(0) = \binom{k}{s}$.

Observe that G is a Cayely graph for the group $\mathbb{Z}_2^k$ with generator set $\{g \in \mathbb{Z}_2^k : |g| \leqslant \frac{k}{2}\}$. Since $\mathbb{Z}_2^k$ is abelian, the eigenvectors of the graphs are independent of the generators, and are simply the characters of the group written as the vector of their values. Namely, corresponding to each $y \in \mathbb{Z}_2^k$ we have an eigenvector v^y, such that $v_x^y = (-1)^{\langle x, y \rangle}$. For every y, $v_0^y = 1$, so to figure out the eigenvalue corresponding to v^y, we simply need to sum the value of v^y on the neighbors of 0. Note that for $y = 0$ we get the all 1s vector, which corresponds to the largest eigenvalue. So we are interested in y's such that $|y| > 0$. By the first two facts above we have

$$\lambda_y = \sum_{g \in \mathbb{Z}_2^k, |g| \leqslant \frac{k}{2}} (-1)^{\langle y, g \rangle} = \sum_{s=0}^{\frac{k-1}{2}} K_s^{(k)}(|y|) = K_{\frac{k-1}{2}}^{(k-1)}(|y| - 1).$$

By the third fact, this is at most $\binom{k-1}{\frac{k-1}{2}} \approx \frac{2^{k-1}}{\sqrt{k-1}} = o(2^{k-1})$. $\quad\square$

4. Graphs with bounded λ_2

Theorem 4 suggests families of graphs that have linear sphericity. Namely, for $0 < \delta \leqslant \frac{1}{2}$, and $\lambda_2 > 0$, the theorem says that δn-regular graphs with second eigenvalue at most λ_2 have linear sphericity. In this section we characterize such graphs. We prove that for $\delta = \frac{1}{2}$ such graphs are nearly complete bipartite, and that for other values, only finitely many graphs exist.

It is worth noting that graphs with bounded second eigenvalue have been previously studied. The apex of these works is probably that of Cameron, Goethals, Seidel and Shult, who characterize in [4] graphs with second eigenvalue at most 2.

4.1. n/2-Regular graphs

In this section we consider the family $\mathbb{G}$ of $n/2$-regular graphs, and second largest eigenvalue λ_2 bounded by a constant. We prove that, asymptotically, they are nearly complete bipartite.

Definition 4.1. Let G and H be two graphs on n vertices. We say that G and H are *close*, if there is a labeling of their vertices such that $|E(G) \triangle E(H)| = o(n^2)$.

Theorem 5. *Every $G \in \mathbb{G}$ is close to $K_{n/2,n/2}$, where n is the number of vertices in G.*

Note 3. *By applying the theorem to the complement graph, if $\lambda_n = O(1)$, then G is close to the disjoint union of two cliques, $K_{n/2} \dot\cup K_{n/2}$.*

We need several lemmas. The first is the well-known Expander Mixing Lemma (cf. [2]). The second is a special case of Simonovitz's Stability Theorem [22], for which we give a simple proof here. The third is a commonly used corollary of Szemeredi's Regularity Lemma. We shall also make use of the Regularity Lemma itself (see e.g. [6]).

Lemma 8. *Let G be an $\frac{n}{2}$-regular graph on n vertices with second largest eigenvalue λ_2. Then every subset of vertices with k vertices has at most $\frac{1}{4}k^2 + \frac{1}{2}\lambda_2 k$ internal edges.*

Lemma 9. *Let R be a triangle-free graph on n vertices, with $n^2/4 - o(n^2)$ edges. Then R is close to $K_{n/2,n/2}$. Furthermore, all but $o(n)$ of the vertices have degree $\frac{n}{2} \pm o(n)$.*

Proof. Denote by d_i the degree of the ith vertex in R, and by m the number of edges. Then

$$\sum_{(i,j)\in E(R)} (d_i + d_j) = \sum_{i\in V(R)} d_i^2 \geqslant \frac{1}{n}\left(\sum_{i\in V(R)} d_i\right)^2 = \frac{4m^2}{n}.$$

Thus, there is some edge $(i, j) \in E(R)$ such that $d_i + d_j \geqslant \frac{4m}{n} = n - o(n)$. Let Γ_i and Γ_j be the neighbor sets of i and j. Since i and j are adjacent, and R has no triangles, the sets Γ_i and Γ_j are disjoint and independent. If we delete the $o(n)$ of vertices in $V\setminus(\Gamma_i \cup \Gamma_j)$ we obtain a bipartite graph. We have deleted only $o(n^2)$ edges, so the remaining graph still has $n^2/4 - o(n^2)$ edges. But this means that $|\Gamma_i|, |\Gamma_j| = \frac{n}{2} - o(n)$, and that the degree of each vertex in these sets is $\frac{n}{2} \pm o(n)$.   □

Recall that the Regularity Lemma states that for every $\varepsilon > 0$ and $m \in \mathbb{N}$ there is an M, such that the vertex set of every large enough graph can be partitioned into k subsets, for some $m \leqslant k \leqslant M$ with the following properties: All subsets except one, the "exceptional" subset, are of the same size. The exceptional subset contains less than an ε-fraction of the vertices. All but an ε-fraction of the pairs of subsets are ε-regular.

The regularity graph with respect to such a partition and a threshold d, has the k subsets as vertices. Two subsets, U_1 and U_2 are adjacent, if they are ε-regular, and $e(U_1, U_2) > d|U_1|^2 = d|U_2|^2$.

Lemma 10 (*Diestel [6], Lemma 7.3.2*). *Let G be a graph on n vertices, $d, \varepsilon \in (0, 1]$, and s be s.t. $\varepsilon \leqslant \frac{(d-\varepsilon)^{2s}}{2s+1}$. Let R be an ε-regularity graph of G, with (non-exceptional) sets of size at least $\frac{s}{\varepsilon}$, and threshold d. If R contains a triangle, then G contains a complete tripartite subgraph, with each side of size s.*

Corollary 2. *If $G \in \mathbb{G}$, and R is as in the lemma, with $s = 10\lambda_2$, then R is triangle free. In this case, if R has $\frac{k^2}{4} - o(k^2)$ edges, then R is close to complete bipartite.*

Proof. If R contains a triangle, then G contains a complete tripartite subgraph, with s vertices on each side. Let U be the set of vertices in this subgraph. Then $e(U) = 3s^2 = 300\lambda_2^2$, but by Lemma 8, $e(U) \leqslant 250\lambda_2^2$, a contradiction. The second part now follows from Lemma 9.   □

Proof of Theorem 5. We would like to apply the Regularity Lemma to graphs in $\mathbb{G}$, and have $\varepsilon = o(1)$, and $k = \omega(1)$ as well as $k = o(n)$. Indeed, this can be done. Since M depends only on m and ε, choose $d = o(1)$, and $m = \omega(1)$, such that the M given by the

lemma satisfies $\frac{n}{(M+1)} \geqslant \frac{s}{\varepsilon}$. As M depends only on m and ε, $\frac{M}{\varepsilon}$ can be made small enough, even with the requirements on d and m.

Let R be the regularity graph for the partition given by the Regularity Lemma, with threshold d as above. Denote by k the number of sets in the partition, and their size by l (so $k \cdot l = n(1 - \eta)$, for some $\eta \leqslant \varepsilon$). We shall show that R is close to complete bipartite, and that G is close to the graph obtained by replacing each vertex in R with l vertices, and replacing each edge in R by a $K_{l,l}$.

Call an edge in G (i) "irregular" if it belongs to an irregular pair; (ii) "internal" if it connects two vertices within the same part; (iii) "redundant" if it belongs to a pair of edge density smaller than d, or touches a vertex in the exceptional set. Otherwise (iv), call it "good".

Recall that $\varepsilon = o(1)$, so only $o(k^2)$ pairs of sets are not ε-regular. Thus, G can have only $o(l^2 k^2) = o(n^2)$ irregular edges. Also, $d = o(1)$, so the number of redundant edges is $k^2 \cdot o(l^2) + o(l)\frac{n}{2} = o(n^2)$. Finally, the number of internal edges is at most $\frac{1}{2}l^2 k$, hence there are $\frac{n^2}{4} - o(n^2)$ good edges.

The number of edges between two sets is at most l^2, so R must have at least

$$\frac{n^2 - o(n^2)}{4l^2} = \frac{k^2}{4} - o(k^2)$$

edges. The corollary implies that it is close to complete bipartite. By Lemma 9, the degree of all but $o(k)$ of the vertices in R is indeed $\frac{k}{2} \pm o(k)$. This means that every edge in R corresponds to $l^2 - o(l^2)$ good edges in G (as the number of edges in R is also no more than $\frac{k^2}{4} + o(k^2)$).

To see that G is close to complete bipartite, let us count how many edges need to be modified. First, delete $o(n^2)$ edges that are not "good". Next, add all possible $o(n^2)$ new edges between pairs of sets that have "good" edges between them. As R is close to complete bipartite, we need to delete or add all edges between $o(k^2)$ pairs. Each such step modifies l^2 edges, altogether $o(l^2 k^2) = o(n^2)$ modifications. Finally, divide the $o(n)$ vertices of the exceptional set evenly between the two sides of the bipartite graph, and add all the required edges, and the tally remains $o(n^2)$. $\quad\square$

Note 4. *In essence, the proof shows that a graph with no dense induced subgraphs is close to complete bipartite. This claim is similar in flavor to Bruce Reed's Mangoes and Blueberries theorem* [18]. *Namely, that if every induced subgraph G' of G has an independent set of size $\frac{1}{2}|G'| - O(1)$, then G is close to being bipartite. The conclusion in Reed's theorem is stronger in that only a linear number of edges need to be deleted to get a bipartite graph.*

Note 5. *In fact, the proof gives something a bit stronger. Let $t_r(n)$ be the number of edges in an n-vertex complete r-partite graph, with parts of equal size. Using the general Stability Theorem* [22] *instead of Lemma 9, the same proof shows that if a graph has $t_n - o(n^2)$ edges and no dense induced subgraphs, then it is close to being complete r-partite.*

4.2. δn-Regular graphs

In Theorem 5 we required that the degree is $n/2$. We can deduce from the theorem that this requirement can be relaxed:

Corollary 3. *Let* $\mathbb{G}$ *be a family of d-regular graphs, with* $d \leqslant \frac{n}{2}$ *(n being the number of vertices in the graph) and bounded second eigenvalue, then every* $G \in \mathbb{G}$ *is close to a complete bipartite graph.*

Proof. Let $M \in \mathbb{M}_n$ be the adjacency matrix of such a d-regular graph, and denote $\bar{M} = J - M$, where J is the all ones matrix. Consider the graph H corresponding to the following matrix:

$$N = \begin{pmatrix} M & \bar{M} \\ \bar{M}^t & M \end{pmatrix}.$$

Clearly H is an n-regular graph on $2n$ vertices. Denote by (x, y) the concatenation of two n-dimensional vectors, x, y, into a $2n$-dimensional vector. Let v be an eigenvector of M corresponding to eigenvalue λ. It is easy to see that v is also an eigenvalue of $\bar{M}$: If $v = \vec{1}$ (and thus $\lambda = d$) it corresponds to eigenvalue $n - \lambda$, otherwise to $(-\lambda)$.

Thus, (v, v) and $(v, -v)$ are both eigenvectors of N. If $v = \vec{1}$ they correspond to eigenvalues n, $2d - n$, respectively, otherwise to 0, 2λ. Since the v's are linearly independent, so are the $2n$ vectors of the form (v, v) and $(v, -v)$: Consider a linear combination of these vectors that gives 0. Both the sum and the difference of the coefficients of each pair have to be 0, and thus both are 0. So we know the entire spectrum of N, and see, since $d \leqslant \frac{n}{2}$, that Theorem 5 holds for it.

Let H' be a complete bipartite graph that is close to H. Since H differs from H' by $o(n^2)$ edges, the same holds for subgraphs over the same set of vertices. In particular, G is close to the subgraph of H' spanned by the first n vertices. Obviously, every such subgraph is itself complete bipartite. $\square$

Corollary 4. *For every* $0 < \delta < \frac{1}{2}$ *and c, there are only finitely many δn-regular graphs with* $\lambda_2 < c$.

Proof. Consider such a graph with n large. By the previous corollary it is close to complete bipartite. Since it is also regular, it must be close to $K_{\frac{n}{2}, \frac{n}{2}}$, which contradicts the constraint $\delta < \frac{1}{2}$. $\square$

4.3. Graphs with both λ_2 and λ_{n-1} bounded by a constant

Theorem 5 can loosely be stated as follows: A regular graph with spectrum similar to that of a bipartite graph (λ_1 being close to $n/2$ and λ_2 being close to 0) is close to being complete bipartite. We conclude this section by noting that if we strengthen the assumption on how close the spectrum of a graph is to that of a bipartite graph, we get a stronger result as to how close it is to a complete bipartite graph.

Theorem 6. *Let $\mathbb{G}$ be a family of $\frac{n}{2}$-regular graphs on n vertices, with both λ_2 and λ_{n-1} bounded by a constant. Then every $G \in \mathbb{G}$ is close to a $K_{\frac{n}{2}, \frac{n}{2}}$, in the sense that such a graph can be obtained from G by modifying a linear number of edges for $O(\sqrt{n})$ vertices of G, and $O(\sqrt{n})$ edges for the rest.*

Proof. First note that it follows that $\lambda_n(G) = -\frac{n}{2} + O(1)$. Take $G \in \mathbb{G}$, and let A be its adjacency matrix. Clearly $tr(A^2) = \frac{n^2}{2}$. If $\lambda_{n-1}(G) = -O(1)$, then

$$\frac{n^2}{2} = tr(A^2) = \lambda_1^2 + \lambda_n^2 + \sum_{i=2,\dots,n-1} \lambda_i^2.$$

Since $\lambda_1 = \frac{n}{2}$

$$\lambda_n^2 = \frac{n^2}{2} - \left(\frac{n}{2}\right)^2 - \sum_{i=2,\dots,n-1} \lambda_i^2.$$

As $\lambda_2, \dots, \lambda_{n-1} = O(1)$ we have

$$\lambda_n^2 = \frac{n^2}{4} + O(n).$$

And since λ_n is negative, and is smaller than λ_1 in absolute value:

$$\lambda_n = -\frac{n}{2} + O(1).$$

Let x be an eigenvector corresponding to λ_n. Suppose, w.l.o.g. that $\|x\|_\infty = 1$ and that $x_v = 1$. Denote $A = \{u : x_u \leqslant -(1 - \frac{1}{\sqrt{n}})\}$, and $B = \{w : x_w \geqslant (1 - \frac{1}{\sqrt{n}})\}$. The eigenvalue condition on v entails

$$\frac{n}{2} - O(1) = -\sum_{u:(u,v)\in E} x_u.$$

Thus, there is a vertex u such that $x_u \leqslant -(1 - O(\frac{1}{n}))$. It is not hard to verify that v must have $\frac{n}{2} - O(\sqrt{n})$ neighbors in A, and that u must have $\frac{n}{2} - O(\sqrt{n})$ neighbors in B.

Now denote $A' = \{u : x_u \leqslant -\frac{1}{2}\}$, and $B' = \{w : x_w \geqslant \frac{1}{2}\}$. Again, it is not hard to check that each vertex in A must have $\frac{n}{2} - O(\sqrt{n})$ neighbors in B', and vice versa. Thus, delete the $O(\sqrt{n})$ vertices that are neither in A nor in B. For each remaining vertex in A (similarly in B), its degree is at most $\frac{n}{2}$, and at least $\frac{n}{2} - O(\sqrt{n})$. It has $\frac{n}{2} - O(\sqrt{n})$ neighbors in B, so the number of its neighbors in A, and the number of its non-neighbors in B is $O(\sqrt{n})$. By deleting and adding $O(\sqrt{n})$ edges to each vertex, we get a complete bipartite graph. $\square$

Note 6. *Alternatively, we could have defined $\mathbb{G}$ as a family of $\frac{n}{2}$-regular graphs with λ_2 bounded, and $\lambda_n(G) = -\frac{n}{2} + O(1)$. It's interesting to note that in this case it follows that λ_{n-1} is bounded. For $G \in \mathbb{G}$, if G is bipartite, then it is complete bipartite, and $\lambda_{n-1}(G) = 0$. Otherwise, $\chi(G) > 2$, and by a theorem of Hoffman [9] $\lambda_n(G) + \lambda_{n-1}(G) + \lambda_1(G) \geqslant 0$. By our assumption, $\lambda_n(G) + \lambda_1(G) = O(1)$, and since $\lambda_{n-1}(G) < 0$ (otherwise the eigenvalues won't sum up to 0), it follows that $\lambda_{n-1}(G) = -O(1)$.*

5. Conclusion and open problems

The only explicit examples known so far for graphs that have linear sphericity are $K_{n,n}$ and small modifications of it. We conjecture that more complicated graphs, such as the Paley graph, also have linear sphericity. Note that the lower bound presented here only shows a bound of $\Omega(\sqrt{n})$. It is also interesting to know if the bound can be improved, either as a pure spectral bound, or with some further assumptions on the structure of the graph.

What is the largest sphericity, $d = d(n)$, of an n-vertex graph? We know that $\frac{n}{2} \leqslant d \leqslant n-1$. Can this gap be closed? For a seemingly related question, the smallest dimension required to realize a sign matrix (see [1]) the answer is known to be $\frac{n}{2} \pm o(n)$. We have also seen a similar gap for $d(n, l_2)$ and $d(n, l_\infty)$. Can this be closed? Can some kind of interpolation arguments generalize the bounds we know for these two numbers to bounds on $d(n, l_p)$ for $p > 2$?

Finally, we have seen that $\frac{n}{2}$-regular graphs with bounded second eigenvalue are $o(n^2)$-close to complete bipartite. However, the only example we know of such graphs are constructed by taking a complete bipartite graph, and changing a constant number of edges for each vertex. These graphs are $O(n)$-close to being complete bipartite. Are there examples of such families which are further from complete bipartite graphs, or can a stronger notion of closeness be proved?

Acknowledgments

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