Discrete Mathematics 58 (1986) 97–98
North-Holland

NOTE

GRAPH COLORING AND MONOTONE FUNCTIONS ON POSETS

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Received 13 December 1984
Revised 22 May 1985

The purpose of this note is to point out a relationship between graph coloring and monotone functions defined on posets. This relationship permits us to deduce certain properties of the chromatic polynomial of a graph.

Theorem 1. *Let $G = (V, E)$ be a graph of order p and $\chi_G(x) = \chi(x)$ its chromatic polynomial. The generating function*

$$f(x) = f_G(x) = \sum_{n=1}^{\infty} \chi(n)x^n$$

is a rational function of the form

$$f(x) = \frac{Q(x)}{(1-x)^{p+1}},$$

where Q is a polynomial with nonnegative coefficients of degree p with leading coefficient $a(G)$, the number of acyclic orientation of G.

Proof. We denote $V = [p] = \{1, \ldots, p\}$, $A(G)$ is the set of acyclic orientations of G and $a(G) = |A(G)|$ is their number. An n-coloring of G, $c: V \to [n]$ induces an acyclic orientation $D_c \in A(G)$ as follows: If $[x, y] \in E$ is an edge, where $c(x) > c(y)$ then in D_c this edge is oriented from x to y. Every acyclic orientation $D \in A(G)$ defines a partial order on V, which we denote by $\geqslant_D$. If $D \in A(G)$, then we think of D as both an acyclic orientation and as a partial order on V.

Note that for an n-coloring $c: V \to [n]$ the function c is a strongly order-preserving map from the poset $(V, \geqslant_D)$ into $[n]$. It is easily verified that the correspondence between n-colorings and strong order preserving maps from acyclic orientations into $[n]$, is bijective. For a poset $(P, \geqslant)$ we let μ_P be its strong order polynomial, namely, $\mu(n) = \mu_P(n)$ is the number of strongly monotone

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functions $(P, \geqslant) \to [n]$. Our basic observation is thus that

$$\chi(x) = \sum_{D \in A(G)} \mu_D(x), \tag{1}$$

where $\chi(x) = \chi_G(x)$ is the chromatic polynomial of G.

Now in [1] it is shown that for a poset D

$$\sum_{n=1}^{\infty} \mu_D(n) x^n$$

is a rational function of the form $R(x)/(1-x)^{p+1}$ where R is a polynomial of degree p with nonnegative coefficients and with leading coefficient 1. Using (1) the conclusion follows. □

In [1] the coefficients of $R(x)$ are interpreted in terms of combinatorial properties of the poset D. Except for the leading coefficient of Q which is $a(G)$ we do not have any relations between the coefficients of Q and other parameters of G. It may be worthwhile to find if such relations exist.

Reference

[1] R. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc. 119 (1972).