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Largest Induced Suborders Satisfying the Chain Condition

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Abstract. For a finite ordered set P, let $c(P)$ denote the cardinality of the largest subset Q such that the induced suborder on Q satisfies the Jordan–Dedekind chain condition (JDCC), i.e., every maximal chain in Q has the same cardinality. For positive integers n, let $f(n)$ be the minimum of $c(P)$ over all ordered sets P of cardinality n. We prove: $\sqrt{2n} - 1 \leqslant f(n) \leqslant 4e\sqrt{n}$.

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For a finite ordered set P, let $c(P)$ denote the size of the largest subset Q such that Q with the induced order satisfies the Jordan–Dedekind chain condition (JDCC), i.e., every maximal chain in Q has the same cardinality. We consider the question, in terms of the cardinality of P, how small can $c(P)$ be? For positive integers n, let $f(n)$ be the minimum of $c(P)$ over all ordered sets P of cardinality n. $f(n)$ is trivially at least $\sqrt{n}$, since every ordered set on n elements has either a chain or antichain of cardinality $\sqrt{n}$ (a simple consequence of Dilworth's theorem [1]). We prove:

THEOREM 1. $\sqrt{2n} - 1 \leqslant f(n) \leqslant 4e\sqrt{n}$, *for all n.*

Before giving the proof, let us mention that this is an example of a wide variety of questions of the form: given a 'nice' class of ordered sets and given an arbitrary ordered set P, how large an induced suborder must P have that belongs to this class. Another such question (to which we do not know any nontrivial bounds) arises if we take the class to be the ordered sets of dimension two. Similar questions have been studied in the context of graphs and we believe it to be worthwhile to consider such questions for ordered sets.

The remainder of this paper is devoted to proving Theorem 1. The lower bound of the theorem is a strengthening of the observation which gave the $\sqrt{n}$ lower bound. Let A be a largest antichain of P and let B be a largest antichain of P–A. Then every element of B is related to some element of A (otherwise A is not maximal). Let A' be the subset of A consisting of all elements that are related to some element of B; $|A'| \geqslant |B|$, otherwise $B \cup (A-A')$ is a larger antichain than A. Hence, $B \cup A'$ is a height one poset with no isolated elements (thus satisfying JDCC) having cardinality at least $2|B|$. If $2|B| \geqslant \sqrt{2n}-1$ or $|A| \geqslant \sqrt{2n}-1$ then $c(P) \geqslant \sqrt{2n}-1$ so suppose $|A| < \sqrt{2n}-1$ and $|B| < (\sqrt{n}/\sqrt{2}) - \frac{1}{2}$. Then P–A has cardinality greater than $n-\sqrt{2n}+1$ and no antichain of size $(\sqrt{n}/\sqrt{2}) - \frac{1}{2}$, so it has a chain of size $(n-\sqrt{2n}+1)/(\sqrt{n/2}-\frac{1}{2}) \geqslant \sqrt{2n}-1$, which completes the proof of the lower bound.

We now proceed to the upper bound. The proof we present is nonconstructive; we show that for every n there exists a poset on n elements so that every subposet of size greater than $4e\sqrt{n}$ does not satisfy JDCC.

Let σ be a permutation of $[n]$ $(=\{1, 2, \ldots, n\})$. The *permutation order* $P(\sigma)$ associated with σ is defined on the set $\{x_1, \ldots, x_n\}$ by $x_i < x_j$ if $i<j$ and $\sigma(i)<\sigma(j)$. (The class of permutation orders is equivalent to the class of two-dimensional orders and has been studied extensively.) It will be convenient to view a permutation σ as a sequence $\sigma(1), \sigma(2), \ldots, \sigma(n)$. By a subsequence of σ we mean a sequence $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)$ where $i_1 < i_2 < \cdots < i_k$. Observe that the induced suborders of $P(\sigma)$ are associated to subsequences of σ and are themselves permutation orders. We will say that a permutation σ (or more generally, a subsequence $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)$) is JDCC if the associated permutation order (or induced suborder) satisfies JDCC. We let $g(n)$ denote the number of permutations of $[n]$ satisfying JDCC.

We will prove

THEOREM 2. *For any $m > 4e\sqrt{n}$ there exists a permutation order on n elements such that any induced suborder of size m or greater fails to satisfy JDCC, and therefore $f(n) \leqslant 4e\sqrt{n}$.*

The proof of Theorem 2 is based on two lemmas.

LEMMA 3. *If for every permutation σ of n, $P(\sigma)$ has an induced suborder of size at least m satisfying JDCC then:*

$$\sum_{j=m}^{2m-2} \frac{g(j)}{j!} \binom{n}{j} \geqslant 1.$$

LEMMA 4. *For all m, $g(m) \leqslant 16^m$.*

To see that Lemmas 3 and 4 imply Theorem 2, suppose that every permutation order of size n has an induced suborder satisfying JDCC of size at least m. Then by Lemmas 3 and 4:

$$\sum_{j=m}^{2m-2} \frac{16^j}{j!} \binom{n}{j} \geqslant 1.$$

Using the inequalities $n!/(n-j)! \leqslant n^j$ and $j! \geqslant j^{j+1/2}/e^j$, we obtain:

$$\sum_{j=m}^{2m-2} \frac{1}{j} \left[\frac{16e^2 n}{j^2}\right]^j \geqslant 1.$$

If $m \geqslant 4e\sqrt{n}$, each summand is less than $1/m$ and the sum is less than $(m-1)/m$, a contradiction establishing $f(n) < 4e\sqrt{n}$.

Proof of Lemma 3. We begin with a

Claim. If P has an induced suborder of size at least m satisfying JDCC, then it has such an induced suborder of size between m and $2m-2$.

To see this, let Q be the smallest induced suborder of P satisfying JDCC and having size at least m. Assume $|Q| > m$; by minimality of Q it is not an antichain. Let M be the set of minimal elements of Q. Then $Q \neq M$ and both M and Q–M satisfy JDCC, so by the choice of Q, $|M| < m$ and $|Q\text{–}M| < m$, and so $|Q| \leqslant 2m-2$. This establishing the claim.

Now let σ be a random permutation of n elements (each permutation is chosen with probability $1/n!$). For a given index set $i_1 < i_2 < \cdots < i_j$ the probability that the induced suborder corresponding to $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_j)$ satisfies JDCC is $g(j)/j!$. Thus the probability that σ has a subsequence of size j satisfying JDCC is at most $\binom{n}{j} g(j)/j!$. If each permutation order of size n has an induced suborder of size at least m satisfying JDCC, then by the above claim, a random permutation contains a subsequence satisfying JDCC of size between m and $2m-2$ with probability 1. Hence

$$\sum_{j=m}^{2m-2} \frac{g(j)}{j!} \binom{n}{j} \geqslant 1.$$

Proof of Lemma 4. To bound $g(n)$, we first need to characterize permutations satisfying JDCC. Note that a chain in $P(\sigma)$ corresponds to an increasing subsequence of σ. Define $c(i;\sigma)$ to be the length of the longest increasing subsequence of σ ending with $\sigma(i)$, that is, $c(i;\sigma) = \max\{k \mid \text{there exists } i_1 < i_2 < \cdots < i_k = i \text{ such that } \sigma(i_1) < \cdots < \sigma(i_k)\}$.

Note that an increasing subsequence in σ in positions $i_1 < i_2 < \cdots < i_k$ corresponds to an increasing subsequence in σ^{-1} in positions $\sigma(i_1) < \cdots < \sigma(i_k)$. In particular, we have

$$c(i;\sigma) = c(\sigma(i);\sigma^{-1}). \qquad (*)$$

The function $c(i;\sigma)$ plays a key role in work of Shensted [4] and Greene [3] concerning Young tableau and permutations. The following lemma is implicit in their work.

LEMMA 5. *Every permutation σ is determined uniquely by the functions $c(i;\sigma)$ and $c(i,\sigma^{-1})$.*

Proof. Given the functions $c(i;\sigma)$ and $c(i;\sigma^{-1})$, we show how to reconstruct σ. Fix $k \leqslant n$ and let $i_1 < i_2 < \cdots < i_r$ (resp. $j_1 < \cdots < j_s$) be the set of indices i such that $c(i;\sigma) = k$ (resp. $c(i;\sigma^{-1}) = k$). By $(*)$, σ maps the set $\{i_1, \ldots, i_r\}$ bijectively to the set

$\{j_1, \ldots, j_s\}$, so in particular, $r = s$. Moreover, we must have $\sigma(i_r) < \sigma(i_{r-1}) < \cdots < \sigma(i_1)$ since $\sigma(i_t) < \sigma(i_{t+1})$ would imply $c(i_{t+1}; \sigma) > c(i_t; \sigma)$. Thus $\sigma(i_1) = j_r$, $\sigma(i_2) = j_{r-1}$, $\ldots, \sigma(i_r) = j_1$. Doing this for all $k \in [n]$ determines σ uniquely. □

A function α: $[n] \to [n]$ will be said to be *admissible* if $\alpha(1) = 1$ and $\alpha(i) \leqslant \alpha(i-1) + 1$ for $2 \leqslant i \leqslant n$.

LEMMA 6. *If σ is JDCC then $c(i; \sigma)$ and $c(i; \sigma^{-1})$ are both admissible.*

Proof. Suppose σ is JDCC. Clearly $c(1; \sigma) = 1$. $c(i; \sigma) \leqslant c(i-1; \sigma) + 1$ is trivial if $\sigma(i) < \sigma(i-1)$. If $\sigma(i) > \sigma(i-1)$ then $\sigma(i-1)$ and $\sigma(i)$ are contained together in some maximal (hence, maximum) increasing subsequence. The portion of this sequence ending with $\sigma(i)$ has length $c(i; \sigma)$ (otherwise we could construct a larger maximal chain). Hence, $c(i; \sigma) \leqslant c(i-1; \sigma) + 1$ and $c(i; \sigma)$ is admissible. If σ is JDCC then so is σ^{-1} (since $P(\sigma)$ and $P(\sigma^{-1})$ are isomorphic), so $c(i; \sigma^{-1})$ is admissible. □

Thus, by Lemmas 5 and 6, $g(n)$ is bounded by the number of pairs, α, β of admissible functions of $[n]$, i.e., the square of the number of admissible functions, so Lemma 4 follows if we show that there are at most 4^n admissible functions of n. Associate to any function α: $[n] \to [n]$ the function γ given by $\gamma(i) = 1 + \alpha(i) - \alpha(i+1)$ if $1 \leqslant i \leqslant n-1$ and $\gamma(n) = \alpha(n)$. Then $\gamma(1) + \gamma(2) + \cdots + \gamma(n) = n$, α is determined by γ, and α is admissible if and only if γ is nonnegative. Hence, the number of admissible functions equals the number of nonnegative sequences $\gamma(1), \gamma(2), \ldots, \gamma(n)$ which sum to n. Elementary enumeration yields that there are $\binom{2n-1}{n}$, which is less than 4^n. □

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