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INCIDENCE MATRICES OF SUBSETS—A RANK FORMULA*

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Abstract. Let $n \geqq k \geqq l \geqq 0$ be integers, $\mathbb{F}$ a field, and $X = \{1, \cdots, n\}$. $M = M_{n,l,k}$ is an $\binom{n}{l} \times \binom{n}{k}$ matrix whose rows correspond to l-subsets of X, and columns to k-subsets of X. For $L \in X^{(l)}$, $K \in X^{(k)}$ the (L, K) entry of M is 1 if $L \subset K$, 0 otherwise. The problem is to find the rank of M over the field $\mathbb{F}$. We solve the problem for $\mathbb{F} = \mathbb{Z}_2$ and obtain some result on $\mathbb{F} = \mathbb{Z}_3$. The problem originated in extremal set theory and seems to be applicable also for matroids, codes and designs.

Introduction. The following problem was posed by M. Katchalski and M. A. Perles. Given $n \geqq k \geqq l \geqq 0$, integers, let $X = \{1, 2, \cdots, n\}$. Denote by $X^{(k)}$ the family of all subsets of X of cardinality k. A family of k-sets $\mathcal{K} \subset X^{(k)}$ is said to be closed if, for every $L \in X^{(l)}$, $|\{K \in \mathcal{K} | L \subset K\}|$ is never 1. They wanted to know the smallest number $N = N(n, l, k)$ such that if $\mathcal{A} \subset X^{(k)}$ has more than N sets, then it contains a closed subfamily. For $k = l + 1$, their problem was solved by P. Frankl, who showed that in this case $N = \binom{n-1}{l-1}$. In fact he showed that if $\mathcal{A} \subset X^{(l+1)}$, has more than $\binom{n-1}{l-1}$ sets, then there is a family $\mathcal{K} \subset \mathcal{A}$, such that for every $L \in X^{(l)}$, $|\{K \in \mathcal{K} | L \subset K\}|$ is even. Define a matrix M whose rows (columns) are indexed by $X^{(l)}$ (resp. $X^{(l+1)}$). For $L \in X^{(l)}$, $K \in X^{(l+1)}$, the (L, K) entry is 1 if $L \subset K$, 0 otherwise. Frankl's proof is obtained by showing that the rank of this matrix over $\mathbb{Z}_2$ is $\binom{n-1}{l}$.

This raises the general problem: Given $n \geqq k \geqq l \geqq 0$, integers and a field $\mathbb{F}$, define a matrix $M = M_{n,l,k}$ as follows. Let $X = \{1, \cdots, n\}$, then the rows (columns) of M are indexed by $X^{(l)}$ (resp. $X^{(k)}$). For $L \in X^{(l)}$, $K \in X^{(k)}$, the (L, K) entry of M is 1 if $L \subset K$, 0 otherwise. What is the rank of M over the field $\mathbb{F}$? For $\mathbb{F} = \mathbb{Q}$ the answer appears in the literature [1], [2]; it is $\rho(M) = \min\{\binom{n}{l}, \binom{n}{l}\}$, so M has the highest rank possible. In this paper we solve the problem for $\mathbb{F} = \mathbb{Z}_2$ and for $k = l + 1$ over $\mathbb{Z}_3$.

Define a *cycle* to be a family of k-sets such that every l-set is contained in an even number of these k-sets (this is usually done in algebraic topology). The rank formula over $\mathbb{Z}_2$ gives the largest cardinality of a cycle-free subfamily of $X^{(k)}$.

The rank formula over $\mathbb{Z}_2$. Let s be a nonnegative integer; we define $b(s)$ to be the unique set of nonnegative integers S, for which $s = \sum_{x \in S} 2^x$. Of course, b is an injective function. If p, q are integers with $b(p) \supset b(q)$ we simply write $p \supset q$. This defines a partial ordering on the nonnegative integers.

Define $d = k - l$, and let $D = b(d)$. For a function $f: D \to \mathbb{Z}^+$, the nonnegative integers we define $f(D) = \sum_{x \in D} f(x)$.

THEOREM 1. *For $n \geqq k + l$ the rank of $M_{n,l,k}$ over $\mathbb{Z}_2$ is*

$$\sum_{f: D \to \mathbb{Z}^+} (-1)^{f(D)} \binom{n}{l - \sum_{x \in D} f(x) 2^x}.$$

Notation. We denote the matrix $M_{n-p,l-q,k-r}$ by $[p, q, r]$, where p, q, r are nonnegative integers. Also, $[p, q, r]_l$ stands for $M_{n-p,l-q,l-r}$, and $[p, q, r]_k =$

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$M_{n-p,k-q,k-r}\langle p,q\rangle$ is defined to be the sum

$$\sum_{f:D\to Z^+}(-1)^{f(D)}\binom{n-p}{l-q-\sum_{x\in D}f(x)2^x}.$$

Observe that $M_{n,l,k}$ and $M_{n,n-k,n-l}$ are transposed matrices. Therefore, to cover the case $n\leqq l+k$ in Theorem 1, replace l by $n-k$ in the sum formula.

We need some simple observations which we state without proof.

Observation 1.

$$[0,0,0]=\begin{array}{|c|c|}\hline [1,1,1] & 0 \\ \hline [1,0,1] & [1,0,0] \\ \hline \end{array}\,.$$

where the left (right) columns correspond to k-subsets which contain the element 1, (do not contain 1, resp.). The upper (lower) rows are the l-sets containing (not containing) 1.

Observation 2. For $p\leqq q\leqq r$, $M_{n,p,q}\cdot M_{n,q,r}=M_{n,p,r}\cdot\binom{r-p}{r-q}$.

Observation 3. $\binom{a}{b}$ is odd iff $a\supset b$.

Observation 4. $\langle p,q\rangle=\langle p+1,q\rangle+\langle p+1,q+1\rangle$.

Convention. If A is a matrix which depends on n, l, k, then $A(p,q,r)$ denotes the matrix which is obtained by replacing n by $n-p$, l by $l-q$ and k by $k-r$. Similarly, if A depends only on n and l (n and k), then $A(p,q)$ results on replacing n by $n-p$ and l by $l-q(k-q$, resp.).

Let t be a nonnegative integer; then we define

$$S_t=\sum_{j\subset t}\langle t,j\rangle.$$

Also we define a block matrix A_t, indexed by all j such that $j\subset t$. Let $b(t)=\{a_1,\cdots,a_\tau\}$ with $a_1>a_2\cdots>a_\tau\geqq 0$. For $i,j\subset t$ the (i,j) block of A_t is $[t,i,j]$ if $j\supset i$ and $b(j-i)=\{a_1,\cdots,a_\nu\}$ for some $\nu\geqq 0$. All the other blocks are zero. Note that

$$S_0=\langle 0,0\rangle,\qquad A_0=[0,0,0],$$

and so we want to show that $\rho(A_0)=S_0$. Defining α by $2^\alpha\|d$, we prove the stronger:

PROPOSITION 1. *For* $0\leqq t\leqq 2^\alpha$, $\rho(A_t)=S_t$.

Proof. By induction on n. For $n=0,1$ there is nothing to prove. To perform the inductive step, we show that under the induction hypothesis the following hold:

PROPOSITION 2. $\rho(A_{2^\alpha})=S_{2^\alpha}$.

PROPOSITION 3. *For* $0\leqq t\leqq 2^\alpha$, $\rho(A_{t+1})=S_{t+1}$ *implies* $\rho(A_t)=S_t$.

It is clear how Proposition 1 follows from Propositions 2, 3 by a backward induction.

Proof of Proposition 2. For $t=2^\alpha$, $b(t)=\{\alpha\}$, so:

$$A_t=\begin{array}{|c|c|}\hline [t,t,t] & \\ \hline [t,0,t] & [t,0,0] \\ \hline \end{array}\,,\qquad S_t=\langle t,0\rangle+\langle t,t\rangle.$$

The matrices

$$I+\begin{array}{|c|c|}\hline 0 & [t,t,0]_l \\ \hline 0 & 0 \\ \hline\end{array}\,, \qquad I+\begin{array}{|c|c|}\hline 0 & [t,t,0]_k \\ \hline 0 & 0 \\ \hline\end{array}$$

are nonsingular (in fact they are self-inverse), and they satisfy

$$\left(I+\begin{array}{|c|c|}\hline \quad & [t,t,0]_l \\ \hline \quad & \quad \\ \hline\end{array}\right)A_t\left(I+\begin{array}{|c|c|}\hline \quad & [t,t,0]_k \\ \hline \quad & \quad \\ \hline\end{array}\right)=\begin{array}{|c|c|}\hline \quad & \quad \\ \hline [t,0,t] & \quad \\ \hline\end{array}\,.$$

To prove this, use Observations 2, 3 to show that in $\mathbb{Z}_2$ $[t,t,0]_l[t,0,0]=[t,t,0]\cdot\binom{d+t}{t}=0$, since $t=2^\alpha\|d$, and so $(d+t)\not\supset t$. Similarly $[t,t,t][t,t,0]_k=0$. But $[t,t,0]_l[t,0,t]=[t,t,t]\binom{d}{t}=[t,t,t]$, since $d\supset t$, and also $[t,0,t][t,t,0]_k=[t,0,0]$ for the same reason.

Rank is preserved under multiplying by the nonsingular matrices, and so $\rho(A_t)=\rho([t,0,t])$. From the induction hypothesis the last rank is

$$\sum_{f:D\setminus\{\alpha\}\to\mathbb{Z}^+}(-1)^{f(D\setminus\{\alpha\})}\binom{n-t}{l-\sum_{x\in D\setminus\{\alpha\}}f(x)2^x}.$$

Now $S_t=\langle t,0\rangle+\langle t,t\rangle=\sum_{f:D\to\mathbb{Z}^+}(-1)^{f(D)}\left[\binom{n-t}{l-\sum_{x\in D}f(x)2^x}+\binom{n-t}{l-t-\sum_{x\in D}f(x)2^x}\right].$

All the second summands appear also as first summands with the opposite sign: increase $f(\alpha)$ by one. Doing all the canceling, we obtain only the sum of the first terms in which $f(\alpha)=0$; i.e.,

$$\sum_{f:D\setminus\{\alpha\}\to\mathbb{Z}^+}(-1)^{f(D\setminus\{\alpha\})}\binom{n-t}{l-\sum_{x\in D\setminus\{\alpha\}}f(x)2^x},$$

$\rho(A_t)=S_t$ for $t=2^\alpha$. □

Now we turn to the proof of Proposition 3. We establish a relation between A_t and A_{t+1}, between S_t and S_{t+1}. We define λ by $2^\lambda\|(t+1)$.

PROPOSITION 4.

$$S_t=S_{t+1}+2\sum_{\substack{j\subset t\\ 2^\lambda\nmid j}}\langle t+1,j\rangle.$$

PROPOSITION 5.

$$\rho(A_t)=\rho(A_{t+1})+2\sum_{0\leqq\nu<\lambda}\rho(A_{t+1-2^{\nu+1}}(2^{\nu+1},2^\nu,2^\nu)).$$

First we show how Propositions 4, 5 imply Proposition 3. For any $0 \leqq \nu < \lambda$, set $r = t+1-2^{\nu+1}$. Using the inductive hypothesis we use the equality $\rho(A_r) = S_r = \sum_{j \subset r} \langle r, j \rangle$ with n replaced by $n-2^{\nu+1}$, l by $l-2^{\nu}$ and k by $k-2^{\nu}$; i.e. we use

$$\rho(A_r(2^{\nu+1}, 2^{\nu}, 2^{\nu})) = \sum_{j \subset r} \langle r+2^{\nu+1}, j+2^{\nu} \rangle = \sum_{j \subset r} \langle t+1, j+2^{\nu} \rangle = \sum_{\substack{i \subset t \\ 2^{\nu} \| i}} \langle t+1, i \rangle.$$

The last equality follows on setting $i = j+2^{\nu}$. Summing over all $0 \leqq \nu < \lambda$ yields that $\rho(A_{t+1}) = S_{t+1}$ implies $\rho(A_t) = S_t$; i.e., Proposition 4, 5 imply Proposition 3 and thus the main theorem.

We make the following simple observation.

Observation 5. For two nonnegative integers a, b, $a \subset b+1$ holds iff exactly one of the relations $a \subset b$, $a-1 \subset b$ holds.

Proof of Proposition 4. $S_t = \sum_{j \subset t} \langle t, j \rangle$, and by Observation 4 it equals $\sum_{j \subset t} \langle t+1, j \rangle + \langle t+1, j+1 \rangle = \sum_{j \subset t} \langle t+1, j \rangle + \sum_{j-1 \subset t} \langle t+1, j \rangle$. By Observation 5, this equals $\sum_{j \subset t-1} \langle t+1, j \rangle + 2\sum_{j \subset t,\, j-1 \subset t} \langle t+1, j \rangle$. But ($j \subset t$ and $j-1 \subset t$) is equivalent to ($j \subset t$ and $2^{\lambda} \nmid j$). This proves Proposition 4. □

To prove Proposition 5, we apply Observation 1 to each block of A_t. Thus the i row (column) of A_t is replaced now by two rows (columns) which we denote by i, i^*. The i, j blocks of A_t (being $[t, i, j]$ iff $t \supset i$, $t \supset j$, $j \supset i$ and $b(j-i) = \{a_1, \cdots, a_\nu\}$ for some $\nu \geqq 0$) are replaced by

$[t+1, i+1, j+1]$	0
$[t+1, i, j+1]$	$[t+1, i, j]$

.

A zero block is replaced by

0	0
0	0

with the appropriate dimensions. The resulting matrix is called B_t; it is equal to A_t but described in a different way. B_t is, to sum up, a block matrix whose rows and columns are indexed by all i, i^* satisfying $i \subset t$. The only nonzero blocks in B_t are

$$\left.\begin{aligned} &B_t(i,j) = [t+1, i, j] \\ &B_t(i, j^*) = [t+1, i, j+1] \\ &B_t(i^*, j^*) = [t+1, i+1, j+1] \end{aligned}\right\} \text{ iff } \quad j \supset i,\ b(j-i) = \{a_1, \cdots, a_\nu\} \text{ for some } \nu \geqq 0.$$

We want to define nonsingular matrices P_t, Q_t such that in $C_t = P_t B_t Q_t$ the only nonzero blocks are, for $i, j \subset t$,

$$\left.\begin{aligned} &(i \neq 0) \quad C_t(i,j) = [t+1, i, j] \\ &(j \neq t) \quad C_t(i^*, j^*) = [t+1, i+1, j+1] \end{aligned}\right\} \text{ iff } \quad j \supset i,\ b(j-i) = \{a_1, \cdots, a_\nu\} \text{ for some } \nu \geqq 0,$$

$$C_t(0, j) = [t+1, 0, j] \quad \text{iff} \quad b(j) = \{a_1, \cdots, a_\nu\} \text{ for some } \nu \geqq 0 \text{ and } 2^{\lambda} | j,$$

$$C_t(i^*, t^*) = [t+1, i+1, t+1] \quad \text{iff} \quad b(i) = \{a_\nu, \cdots, a_\tau\} \text{ for some } \nu \geqq 1 \text{ and } 2^{\lambda} | (i+1),$$

$$C_t(0, t^*) = [t+1, 0, t+1].$$

The submatrix of C_t spanned by all $j \subset t$ with $2^\nu \| j$, $0 \leqq \nu < \lambda$, is equal to $A_{t+1-2^{\nu+1}}(2^{\nu+1}, 2^\nu, 2^\nu)$. To see this, we set a one-to-one correspondence between all $j' \subset t+1-2^{\nu+1}$ and all $j \subset t$ with $2^\nu \| j$, given by $j = j' + 2^\nu$. This shows the equality between these matrices. Also the submatrix generated by all j^* with $j \subset t$, $2^\nu \| (j+1)$, $0 \leqq \nu < \lambda$, equals $A_{t+1-2^{\nu+1}}(2^{\nu+1}, 2^\nu, 2^\nu)$. Here we correspond $j' \subset t+1-2^{\nu+1}$ to $j = j' + 2^\nu - 1$, $j \subset t$.

The remaining direct summand of C_t is the one indexed by all $j \subset t$ with $2^\lambda | j$, and by all j^* with $j \subset t$, $2^\lambda | (j+1)$. This submatrix is equal to A_{t+1}: Use the correspondence, to $i \subset t+1-2^\lambda$ assign $j = i \subset t$, and to $i \subset t+1$ with $2^\lambda \| i$ assign $j^* = (i-1)^*$ (note that $i-1 \subset t$). This correspondence shows that this submatrix is really equal to A_{t+1}. Thus, if we can find nonsingular matrices P_t, Q_t so that $P_t B_t Q_t = C_t$, then Proposition 5 is established and therefore also the main theorem.

The matrices P_t, Q_t are defined inductively. Reminding the reader that $b(t) = \{a_1, \cdots, a_\tau\}$ with $a_1 > \cdots > a_\tau \geqq 0$, we do the induction on τ. For $\tau = 0$, i.e. $t = 0$,

$$A_0 = [0, 0, 0],$$

$$A_1 = B_0 = C_0 = \begin{array}{|c|c|} \hline [1,1,1] & 0 \\ \hline [1,0,1] & [1,0,0] \\ \hline \end{array}$$

and so P_0, Q_0 are defined to be identity matrices.

In the general case denote 2^{a_1} by δ, and $s = t - \delta$. We define L_t, K_t to be block matrices, indexed by all i, i^* where $i \subset t$. The only nonzero blocks in these matrices are the $(j+\delta, j^*)$ blocks ($j \subset s$), which are $[t+1, j+\delta, j+1]_l$ and $[t+1, j+\delta, j+1]_k$ respectively.

Except for the cases $t = 2^\lambda - 1$, which will be dealt with later, we define

$$P_t = \begin{array}{|c|c|} \hline P_s(\delta, \delta) & \\ \hline & P_s(\delta, 0) \\ \hline \end{array} (I + L_t), \qquad Q_t = (I + K_t) \begin{array}{|c|c|} \hline Q_s(\delta, \delta) & \\ \hline & Q_s(\delta, 0) \\ \hline \end{array}\,.$$

Note that P_t depends on n, l, t only, and Q_t on n, k, t and so $P_s(x, y)(Q_s(x, y))$ results on replacing n by $n - x$ and l by $l - y$ (k by $k - y$), in $P_s(Q_s$ resp.).

To calculate the product $P_t B_t Q_t$ we start by working out

$$(I + L_t) B_t (I + K_t) = B_t + L_t B_t + B_t K_t + L_t B_t K_t.$$

The only nonzero blocks in $L_t B_t$ are $(i+\delta, j^*)$ blocks with $i \subset s$, $j \subset t$, $i \subset j$, $b(j-i) = \{a_1, \cdots, a_\nu\}$, $(0 \leqq \nu \leqq \tau)$. To find out what this block is we have to make the following product:

$$[t+1, i+\delta, i+1]_l [t+1, i+1, j+1] = [t+1, i+\delta, j+1] \cdot \binom{d+i+\delta-j-1}{\delta-1}.$$

The binomial coefficient is odd iff

$$\delta | (d + i - j).$$

We are assuming in Proposition 5 that $t < 2^\alpha$, where $2^\alpha \| d$, so $a_1 < \alpha$ and $\delta | d$. Hence, the condition is equivalent to $\delta | (j - i)$; but $j - i = 2^{a_1} + \cdots + 2^{a_h}$ and this is equivalent to $h = 0, 1$. Therefore, the only nonzero blocks in $L_t B_t$ are: for $j \subset s$ the $(j+\delta, j^*)$ block is $[t+1, j+\delta, j+1]$, and the $(j+\delta, (j+\delta)^*)$ block is $[t+1, j+\delta, j+\delta+1]$.

Similarly, the only nonzero blocks in B_tK_t are: for $j \subset s$, the (j, j^*) block is $[t+1, j, j+1]$ and the $(j+\delta, j^*)$ block is $[t+1, j+\delta, j+1]$. Therefore, in $L_tB_t + B_tK_t$ the only nonzero blocks are: for $j \subset t$, the (j, j^*) block is $[t+1, j, j+1]$.

It is easy to check that $L_tB_tK_t = 0$.

Note that the submatrix of B_t consisting of all $i+\delta$, $(i+\delta)^*$ rows and j, j^* columns with $i, j \subset s$ is equal to $B_s(\delta, 0, \delta)$, and so

$$(I+L_t)B_t(I+K_t) = \Lambda_t + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline B_s(\delta, 0, \delta) & 0 \\ \hline \end{array},$$

where the only nonzero blocks in Λ_t are the (j, j) block $[t+1, j, j]$ and the (j^*, j^*) block $[t+1, j+1, j+1]$ for all $j \subset t$. Note also that $(I+L_t)\Lambda_t(I+K_t) = \Lambda_t$ (details are easy and are omitted) and so in the inductive process of defining P_t, Q_t we have $P_t\Lambda_tQ_t = \Lambda_t$.

By definition of Λ_t

$$\Lambda_t = \begin{array}{|c|c|}\hline \Lambda_s(\delta, \delta, \delta) & \\ \hline & \Lambda_s(\delta, 0, 0) \\ \hline \end{array},$$

and so

$$P_tB_tQ_t = \begin{array}{|c|c|}\hline P_s(\delta, \delta) & \\ \hline & P_s(\delta, 0) \\ \hline \end{array} \left(\Lambda_t + \begin{array}{|c|c|}\hline & \\ \hline B_s(\delta, 0, \delta) & \\ \hline \end{array} \right) \begin{array}{|c|c|}\hline Q_s(\delta, \delta) & \\ \hline & Q_s(\delta, 0) \\ \hline \end{array}$$

$$= A_t + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline C_s(\delta, 0, \delta) & 0 \\ \hline \end{array}.$$

In the last equality we made use of the fact that $P_s\Lambda_sQ_s = \Lambda_s$ and $P_sB_sQ_s = C_s$. It can be checked now that the only nonzero blocks $P_tB_tQ_t$ are given by: for $i, j \subset t$,

$$\left.\begin{array}{ll} (i \neq 0) & P_tB_tQ_t(i, j) = [t+1, i, j] \\ (j \neq t) & P_tB_tQ_t(i^*, j^*) = [t+1, i+1, j+1] \end{array}\right\} \text{ iff } \begin{array}{l} j \supset i,\ b(j-i) = \{a_1, \cdots, a_\nu\} \\ \text{for some } \nu \geqq 0, \end{array}$$

$$P_tB_tQ_t(0, j) = [t+1, 0, j] \quad \text{iff} \quad b(j) = \{a_1, \cdots, a_\nu\} \text{ with } \nu \geqq 0,\ 2^\mu | j,$$

$$P_tB_tQ_t(i^*, t^*) = [t+1, i+1, t+1] \quad \text{iff} \quad b(i) = \{a_\nu, \cdots, a_\tau\} \text{ with } \nu \geqq 1,\ 2^\mu | (i+1),$$

$$P_tB_tQ_t(0, t^*) = [t+1, 0, t+1],$$

where μ is defined by $2^\mu \| (s+1)$.

Since we assumed that t is different from $2^\lambda - 1$, it follows that $\mu = \lambda$, and so $P_tB_tQ_t = C_t$ as we wanted.

So assume $t = 2^\lambda - 1$ and so $\mu = \lambda - 1$ and $s = 2^\mu - 1$. In this case we define X_t (resp. Y_t) as we define P_t(resp. Q_t) in the general case. The only way $X_tB_tY_t$ differs from

C_t in this case is that it has the added nonzero $(0, j)$ blocks with $b(j)=\{a_1, \cdots, a_\nu\}$, $\nu \geqq 0$ and $2^\mu \| j$ and the (i^*, t^*) blocks with $b(i)=\{a_\nu, \cdots, a_\tau\}$ with $\nu \geqq 1, 2^\mu \|(i+1)$. The only block of the first kind is the $(0, \delta)$ block which equals $[t+1, 0, \delta]$ and of the second kind, the (s^*, t^*) block, being $[t+1, s+1, t+1]$.

We define the matrices E_t(resp. F_t) as block matrices indexed by all i, i^* with $i \subset t$, and the only nonzero block being the $(s^*, 0)$ block which equals $[2^\lambda, 2^\mu, 0]_l$ (resp.$[2^\lambda, 2^\mu, 0]_k$). We define $P_t=(I+E_t)X_t$ and $Q_t=Y_t(I+F_t)$ and check that $P_tB_tQ_t=C_t$, as desired. This completes the proof of the main theorem.

A rank formula over $\mathbb{Z}_3$.

THEOREM 2. *The rank of $M_{n,l,l+1}$ over $\mathbb{Z}_3$ is*

$$\sum_{j\geqq 0}\binom{n-2j-1}{l-j}.$$

For $n \geqq 2l+1$ this equals

$$\sum_{j\geqq 0}\binom{n}{l-3j}-\sum_{j\geqq 0}\binom{n}{l-3j-2}.$$

Proof. Let F be a set of nonnegative integers; then we set $w(F)=\sum_{x\in F} 2^x$, (of course, $w=b^{-1}$). Let $X=\{1, \cdots, n\}$ be our base set. We show that $\mathcal{F}=\{F\in X^{(l)} | w(F)<2^{n+2}/3\}$ is an independent set of rows. Since $\mathcal{F}=\{F\in X^{(l)} | n\notin F\}\cup\{F\in X^{(l)} | n\in F,\ (n-1)\notin F,\ (n-2)\notin F\}\cup\{F\in X^{(l)} | n\in F,\ (n-1)\notin F,\ (n-2)\in F, (n-3)\notin F, (n-4)\notin F\}\cup\cdots$, and this union is a disjoint union, $|\mathcal{F}|=\sum_{j\geqq 0}\binom{n-2j-1}{l-j}$ and this shows that the rank is at least this big. We prove that $\mathcal{F}$ is an independent set of rows by induction on n. For any n and $l=0$, $n-1$ this is clear. To perform the inductive step, define $Y=\{1, \cdots, n-2\}$,

$$\mathcal{B}_1=\{B\in Y^{(l-1)} | w(B)<2^n/3\},$$

$$\mathcal{B}_2=\{B\in Y^{(l-1)} | w(B)>2^n/3\}.$$

If $\mathcal{F}$ is dependent, this means that there is a function $f:\mathcal{F}\to\mathbb{Z}_3$, so that

$$\forall A\in X^{(l+1)} \quad \sum_{\substack{F\subset A\\ F\in\mathcal{F}}} f(F)=0$$

For $B\in\mathcal{B}_2$, let $A=B\cup\{n-1, n\}$, to obtain

$$f(B\cup\{n-1\})=0 \quad \forall B\in\mathcal{B}_2.$$

For $B\in\mathcal{B}_1$, $A=B\cup\{n-1, n\}$ we get $f(B\cup\{n-1\})+f(B\cup\{n\})=0$.

For $C\in Y^{(l)}$, let $A=C\cup\{n\}$; then we get

$$\forall C\in Y^{(l)} \quad f(C)+\sum_{\substack{B\subset C\\ B\in\mathcal{B}_1}} f(B\cup\{n\})=0$$

and for $A=C\cup\{n-1\}$ we have

$$f(C)+\sum_{\substack{B\subset C\\ B\in Y^{(l-1)}}} f(B\cup\{n-1\})=0,$$

$$f(C)+\sum_{\substack{B\subset C\\ B\in\mathcal{B}_1}} f(B\cup\{n-1\})+\sum_{\substack{B\subset C\\ B\in\mathcal{B}_2}} f(B\cup\{n-1\})=0.$$

All these equalities easily imply

$$\forall C \in Y^{(l)} \quad \sum_{\substack{B\subset C\\ B\in \mathcal{B}_1}} f(B\cup\{n\})=0.$$

But this shows that in $M_{n-2,l-1,l}$, where the basic set is Y, the rows of $\mathcal{B}_1=\{B\in Y^{(l-1)}|w(B)<2^n/3\}$ are linearly dependent, and this contradicts the induction hypothesis.

For the reverse inequality we first make:

Observation 6. Let P be a $p\times q$ matrix, Q a $q\times r$ matrix and R an $r\times s$ matrix. If $PQR=0$, then

$$\rho(P)+\rho(Q)+\rho(R)\leqq q+r.$$

Now we prove

$$\rho(M_{n,l,l+1})=\sum_{j\geqq 0}\binom{n-1-2j}{l-j}.$$

For $l\geqq 2$ we have that over $\mathbb{Q}$

$$M_{n,l-2,l-1}\cdot M_{n,l-1,l}\cdot M_{n,l,l+1}=3M_{n,l-2,l+1}$$

so over $\mathbb{Z}_3$,

$$M_{n,l-2,l-1}\cdot M_{n,l-1,l}\cdot M_{n,l,l+1}=0$$

and so over $\mathbb{Z}_3$,

$$\rho(M_{n,l-2,l-1})+\rho(M_{n,l-1,l})+\rho(M_{n,l,l+1})\leqq\binom{n}{l-1}+\binom{n}{l}=\binom{n+1}{l}.$$

The l.h.s. is

$$\geqq\sum_{j\geqq 0}\binom{n-1-2j}{l-2-j}+\binom{n-1-2j}{l-1-j}+\binom{n-1-2j}{l-j}$$
$$=\sum_{j\geqq 0}\binom{n+1-2j}{l-j}-\binom{n-1-2j}{l-1-j}=\sum_{j\geqq 0}\binom{n+1-2j}{l-j}-\sum_{j\geqq 1}\binom{n+1-2j}{l-j}$$
$$=\binom{n+1}{l}.$$

It follows that all inequalities are in fact equalities, which completes the proof of the first assertion.

The proof that for $n\geqq 2l+1$

$$\sum_{j\geqq 0}\binom{n-2j-1}{l-j}=\sum_{j\geqq 0}\binom{n}{l-3j}-\sum_{j\geqq 0}\binom{n}{l-3j-2}$$

is straightforward, by induction on l. This formula was presented just because it resembles the rank formula of Theorem 1.

REFERENCES

[1] W. FOODY AND A. HEDAYAT, *On theory and applications of BIB designs with repeated blocks*, Ann. Statist., 5 (1977), pp. 932–945.

[2] R. GRAHAM, S.-Y. R. LI AND W.-C. LI, *On the structure of t-designs*, this Journal, 1 (1980), pp. 8–14.