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Note

Extending the Greene–Kleitman Theorem to Directed Graphs

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The celebrated Dilworth theorem (*Ann. of Math.* 51 (1950), 161–166) on the decomposition of finite posets was extended by Greene and Kleitman (*J. Combin. Theory Ser. A* 20 (1976), 41–68). Using the Gallai–Milgram theorem (*Acta Sci. Math.* 21 (1960), 181–186) we prove a theorem on acyclic digraphs which contains the Greene–Kleitman theorem. The method of proof is derived from M. Saks' elegant proof (*Adv. in Math.* 33 (1979), 207–211) of the Greene–Kleitman theorem.

INTRODUCTION

All our graph theoretical terminology is standard. A *path* in a directed graph is always a directed simple path. For a path P in a digraph $G = (V, E)$ we let $|P|$ denote the number of vertices in P. We let $d_k = d_k(G)$ be the largest order of a k-colourable subgraph of G. A *cover* of G is a set $\mathscr{M} = \{M_1, \ldots, M_t\}$ of disjoint paths which cover V, i.e., V is the disjoint union $\bigcup_{i=1}^{t} V(M_i)$. We define

$$B_k(\mathscr{M}) = \sum_{i=1}^{t} \min(k, |M_i|),$$

and

$$e_k = e_k(G) = \min B_k(\mathscr{M}),$$

the minimum being taken over all covers $\mathscr{M}$. The result of this note is that for acyclic digraphs G

$$d_k(G) \geqslant e_k(G).$$

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If G is the digraph of a poset, it is very easily verified that $e_k \geqslant d_k$ and so in the case of a poset $d_k = e_k$ holds for all k. We thus have

(a) Dilworth theorem [1], $d_1 = e_1$ for posets.

(b) Greene–Kleitman theorem [4], $d_k = e_k$ for posets.

(c) Gallai–Milgram theorem [3], $d_1 \geqslant e_1$ for all digraphs.

(d) Ours, $d_k \geqslant e_k$ for acyclic digraphs.

Before we state the theorem and prove it, let us relate it to another result of Gallai [2] which states that: *The longest path in G contains at least $\chi(G)$ vertices.* ($\chi(G)$ is the chromatic number of G.) Our result furnishes a quantitative extension of Gallai's theorem for acyclic digraphs: Letting k be $\chi(G) - 1$, in our theorem we have

$$v > d_k \geqslant e_k,$$

which implies the existence of disjoint paths $P_1, \ldots, P_l$ in G for which

$$\sum_{i=1}^{l} (|P_i| - \chi(G) + 1)^+ \geqslant v - d_k > 0.$$

Let us state our theorem now and prove it.

THE THEOREM. *Let $G = (V, E)$ be an acyclic digraph and let d_k be the largest order of a k-colourable subgraph of G. For $\mathscr{M} = \{M_1, \ldots, M_t\}$, a vertex cover of G by paths, we define*

$$B_k(\mathscr{M}) = \sum_{i=1}^{t} \min(k, |M_i|),$$

and we let $e_k = \min B_k(\mathscr{M})$ over all such covers $\mathscr{M}$. Then

$$d_k \geqslant e_k.$$

Proof. We reduce the proof to the Gallai–Milgram theorem [3], which states that $d_1 \geqslant e_1$ holds. To prove the general case, we define a digraph G_k, whose vertex set is $V_k = V \times \{1, \ldots, k\}$ and whose edge set is

$$E_k = \{[(x, i), (x, j)] \mid x \in V, k \geqslant i > j \geqslant 1\}$$
$$\cup \{[(x, i), (y, i)] \mid [x, y] \in E, k \geqslant i \geqslant 1\}.$$

We prove the theorem by showing

$$d_k(G) = d_1(G_k) \geqslant e_1(G_k) \geqslant e_k(G).$$

The equality $d_k(G) = d_1(G_k)$ is evident and the inequality $d_1(G_k) \geqslant e_1(G_k)$ holds by the Gallai–Milgram theorem. The proof of $e_1(G_k) \geqslant e_k(G)$ is based on Saks [7]; see also [6]. For a cover $\mathscr{M}$ of G_k we let $F = F(\mathscr{M})$ be the set of initial vertices in the paths of $\mathscr{M}$. Also,

$$F_i = \{x \in V \mid (x, i) \in F\}.$$

The cover $\mathscr{M}$ induces a cover of $G \times \{k\}$, the kth level of G_k which is isomorphic to G. We consider it as a cover of G and denote it by $\hat{\mathscr{M}}$. We define a class of covers of G_k which we call special covers. To each cover $\mathscr{M}$ of G_k we associate a special cover $\mathscr{N}$ with no more paths than $\mathscr{M}$ has, i.e., $B_1(\mathscr{M}) \geqslant B_1(\mathscr{N})$. We also show that if $\mathscr{N}$ is a special cover, then,

$$B_1(\mathscr{N}) \geqslant B_k(\hat{\mathscr{N}}).$$

Hence for each cover $\mathscr{M}$ of G_k we have

$$B_1(\mathscr{M}) \geqslant B_1(\mathscr{N}) \geqslant B_k(\hat{\mathscr{N}}),$$

which proves

$$e_1(G_k) \geqslant e_k(G).$$

We say that the cover $\mathscr{N}$ of G_k is *special* if $F_i = F_i(\mathscr{N})$ satisfy:

(i) $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_{k-1}$.

(ii) For $x \in F_k \backslash F_{k-1}$ the vertex which follows (x, k) in its path of $\mathscr{N}$ is $(x, k-1)$.

Let us indicate why $B_1(\mathscr{N}) \geqslant B_k(\hat{\mathscr{N}})$ for a special $\mathscr{N}$: Consider the sum $B_k(\hat{\mathscr{N}}) = \sum \min(k, |\hat{N}|)$ over all paths $\hat{N}$ in $\hat{\mathscr{N}}$. The paths which start at vertices $x \in F_k(\mathscr{N}) \backslash F_{k-1}(\mathscr{N})$ consist of exactly one vertex by (ii), and therefore contribute $|F_k(\mathscr{N}) \backslash F_{k-1}(\mathscr{N})|$ to this sum. The other terms are $\leqslant k$ each and so

$$B_k(\hat{\mathscr{N}}) \leqslant |F_k(\mathscr{N}) \backslash F_{k-1}(\mathscr{N})| + k\,|F_k(\mathscr{N}) \cap F_{k-1}(\mathscr{N})|.$$

But $B_1(\mathscr{N}) = |F(\mathscr{N})| \geqslant k\,|F_k(\mathscr{N}) \cap F_{k-1}(\mathscr{N})| + |F_k(\mathscr{N}) \backslash F_{k-1}(\mathscr{N})|$, by (i) and (ii). This proves

$$B_1(\mathscr{N}) \geqslant B_k(\hat{\mathscr{N}}).$$

Now we have to show how, given a cover $\mathscr{M}$ of G_k, we construct a special cover $\mathscr{N}$ with

$$B_1(\mathscr{M}) \geqslant B_1(\mathscr{N}).$$

To this end we define an operation on a cover $\mathcal{M}$: Let $(x, i) \in F(\mathcal{M})$, and let $(y, j) \in V_k$ be adjacent to (x, i). By "switching (y, j) on (x, i)" we refer to the operation where the path which contklains (y, j) is split into two parts at (y, j) and the first part is appended to the beginning of the path which starts at (x, i).

Let us suppose that $\mathcal{M}$ is a non-special cover of G_k, e.g., $(x, i) \in F = F(\mathcal{M})$, but $(x, i-1) \notin F$. If $(x, i-1)$ is the successor of some (x, j) with $j > i$, switch (x, j) on (x, i). If $(x, i-1)$ follows (x, i) and $i < k$, switch $(x, i+1)$ on (x, i). If $(x, i-1)$ follows $(y, i-1)$ switch (y, i) on (x, i).

It is easily verified that this process terminates and eventually a special cover is obtained, without increasing the number of paths. This completes the proof of the theorem.

Open problems:

(a) Is the conclusion of the theorem true for all digraphs?

(b) If the roles of paths and independent sets are changed does the theorem remain true?

(c) Let us call the reader's attention to the following result [5] which contains the Gallai–Milgram theorem:

THEOREM. *Let $G = (V, E)$ be a digraph, 'then it has a cover $\mathcal{M} = \{M_1, \ldots, M_t\}$ by paths so that there exist vertices $\{x_1, \ldots, x_t\}$, $x_i \in M_i$ $(1 \leqslant i \leqslant t)$ and such that $\{x_i \mid 1 \leqslant i \leqslant t\}$ is an independent set of vertices.*

Can this result be extended to contain the main theorem of this note?

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