

Graphs with Few 3-Cliques and 3-Anticliques are 3-Universal

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Abstract: For given integers k, l we ask whether every large graph with a sufficiently *small* number of k -cliques and k -anticliques must contain an induced copy of every l -vertex graph. Here we prove this claim for $k = l = 3$ with a sharp bound. A similar phenomenon is established as well for tournaments with $k = l = 4$. © 2014 Wiley Periodicals, Inc. *J. Graph Theory* 78: 229–238, 2015

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1. INTRODUCTION

We start by recalling the notion of universality.

Definition 1.1. A graph (resp. tournament) is called l -universal if it contains every l -vertex graph (tournament) as an induced subgraph (subtournament).

We next recall the celebrated Erdős-Hajnal conjecture [5] that we reformulate in a somewhat nonstandard form. As usual we denote by $\omega(G)$, $\alpha(G)$ the clique, respectively, anticlique number of the graph G .

Conjecture 1.2. [Erdős-Hajnal] For every integer l there is an $\epsilon > 0$ such that every n -vertex graph G with $\alpha(G), \omega(G) < n^\epsilon$ is l -universal.

The largest size of a transitive subtournament of the tournament T is denoted by $\text{tr}(T)$. The Erdős-Hajnal conjecture for tournaments states:

Conjecture 1.3. For every integer l there is an $\epsilon > 0$ such that every n -vertex tournament T with $\text{tr}(T) < n^\epsilon$ is l -universal.

As shown by Alon et al. [2], these two conjectures are equivalent.

The Erdős-Hajnal conjecture in both its formulations posits that a graph (resp. a tournament) which satisfies a rather mild upper bound on largest clique and anticlique (resp. transitive set) must be l -universal. In this paper we ask the following:

Problem 1.4. For given integers k, l is every large graph with few k -cliques and k -anticliques necessarily l -universal? Similarly, is a large tournament with only few transitive subtournaments of order k necessarily l -universal?

The answer for the graph problem with $k = l = 3$ turns out to be positive, and we derive a sharp bound for this statement. For tournaments, the case of $k = 3$ is trivial, but the range $k \geq 4$ turns out rather interesting. We prove that an upper bound on the number of transitive 4-vertex subtournaments implies 4-universality. As explained in the last section, this line of thought can be developed in numerous additional ways.

We need some definitions and notations which we state in the language of graphs. Their counterparts for tournaments are obvious. For a fixed l -vertex graph H and an arbitrary graph G we denote by $p(H, G)$ the probability that a randomly chosen set of l vertices in G induces a subgraph that is isomorphic to H . Given an integer l , we let \mathcal{H}_l be the list of all $N = N_l$ isomorphism classes of l -vertex graphs. We refer to the vector $\pi_l(G) = (p(H, G))_{H \in \mathcal{H}_l}$ as the l th local profile of the graph G .

Below we use \mathcal{G} to always denote a sequence of graphs G_n with $|V(G_n)| \rightarrow \infty$. If the limit $\lim_n \lambda(G_n)$ exists, where λ is some graph parameter, we denote this limit by $\lambda(\mathcal{G})$. Likewise $\bar{\lambda}(\mathcal{G}) := \limsup_n \lambda(G_n)$ and $\underline{\lambda}(\mathcal{G}) := \liminf_n \lambda(G_n)$.

For each $i = 0, 1, 2, 3$ there is exactly one graph $P_i \in \mathcal{H}_3$ that has i edges, and we denote $p(P_i, \mathcal{G})$ by $p_i(\mathcal{G})$, or simply by p_i when \mathcal{G} is clear from the context. We note that $p_0(G)$ (resp. $p_1(G)$) equals p_3 (resp. p_2) of its complement graph. For example, in our terminology, Goodman's well-known bound [9] takes the form

Theorem 1.5. [Goodman] For every \mathcal{G} there holds $p_0 + p_3(\mathcal{G}) \geq \frac{1}{4}$.

Jacob Fox (personal communication) has observed that the answer to Problem 1.4 is positive for some $l = \Omega(k)$. Namely, he found the following lemma whose proof appears in Section 4.

Proposition 1.6. *If both $p(K_k, \mathcal{G}) < 2^{-\binom{k}{2}} + \epsilon$ and $p(\overline{K}_k, \mathcal{G}) < 2^{-\binom{k}{2}} + \epsilon$ then \mathcal{G} is ck-universal, where $c > 0$ is a universal constant.*

We define $\Pi_l \subset \mathbb{R}^N$ as the set of all points $\pi \in \mathbb{R}^N$ for which there exists a sequence of graphs \mathcal{G} with $\pi_l(\mathcal{G}) = \pi$. It is still a major open question to get a good description of these sets Π_l . In the present article we add some piece to what is known about Π_3 . At this writing even Π_3 is not yet fully understood (but see [11,16]). The state of our knowledge of Π_l for $l \geq 4$ is really very limited, though some work already exists, e.g., [7,8,12,13,17–19]. Much of the recent progress in this area was achieved using Razborov’s flag algebras method. We say that \mathcal{G} is *l-universal* if $\underline{p}(H, \mathcal{G}) > 0$ for every $H \in \mathcal{H}_l$.

Our main result is

Theorem 1.7. *There is a constant $\rho = 0.159181\dots$ such that every \mathcal{G} with $\overline{p}_0(\mathcal{G}), \overline{p}_3(\mathcal{G}) < \rho$ is 3-universal. The bound is tight.*

The number ρ is defined as $\rho = 6\theta^2(1 - 2\theta)$ where $\theta = 0.427373\dots$ is the largest root of $\theta^3 + \theta^2 - \theta + \frac{1}{6} = 0$.

We prove this theorem in Section 2. In Section 3 we state and prove our results for tournaments. In Section 4 we prove Proposition 1.6 and mention several open questions.

2. PROOF OF THEOREM 1.7 FOR GRAPHS

First, note that by Goodman’s theorem 1.5,

$$\overline{p}_0, \overline{p}_3 \leq \rho < \frac{1}{4} \Rightarrow \underline{p}_0, \underline{p}_3 > 0.$$

It remains to prove that $\underline{p}_1, \underline{p}_2 > 0$. By the above-mentioned symmetry between p_1 and p_2 , it suffices to consider only p_2 .

By passing to a subsequence, if necessary, and arguing by contradiction, it suffices to consider only sequences \mathcal{G} with $p_2(\mathcal{G}) = 0$. By the graph removal lemma [1,3], an n -vertex graph G with $p_2(G) = o(1)$ can be made P_2 -free¹ by flipping only $o(n^2)$ edges². Since this changes $p_0(G)$ and $p_3(G)$ by only $o(1)$, we may apply this removal step to all $G \in \mathcal{G}$, and assume that every $G \in \mathcal{G}$ is P_2 -free. But a graph is P_2 -free iff it is a union of vertex disjoint cliques, so these are the only graphs we consider henceforth. Our goal is to prove that $\max(\overline{p}_0, \overline{p}_3) \geq \rho$ for such graphs.

We proceed with a series of reductions which allow us to make the following assumptions:

1. There is a bound on the number of cliques in all $G \in \mathcal{G}$.
2. Both limits $p_0(\mathcal{G})$ and $p_3(\mathcal{G})$ exist.
3. $p_0(\mathcal{G}) = p_3(\mathcal{G})$.

Under these assumptions, the theorem follows from Lemma 2.1 below.

¹Note that P_2 is a 3-vertex path. This notation is not universally accepted, but hopefully no confusion is created.

²Here $o(1)$ means $o_n(1)$. In general, little-oh terms are taken w.r.t. to the order of the graph that tends to infinity.

It suffices to consider n -vertex graphs G with only a bounded number of non-trivial cliques. For let us fix some $\epsilon > 0$ and remove all the edges from every clique of size $< \epsilon n$. This leaves only $< \frac{1}{\epsilon}$ nontrivial cliques in G which now has the desired form. This changes the parameters $p_0(G), p_3(G)$ only by $O(\epsilon)$. By letting $\epsilon \rightarrow 0$ the reduction follows.

Our next reduction is to graphs G with $|p_0(G) - p_3(G)| \leq O(\frac{1}{n})$. Given the additional assumption that $p_0(\mathcal{G})$ exists, this will imply $p_0(\mathcal{G}) = p_3(\mathcal{G})$. Suppose that $p_0(G) - p_3(G) \gg \frac{1}{n}$ for G an n -vertex graph which is the union of vertex-disjoint cliques. We construct another n -vertex graph G' with $p_0(G) > p_0(G'), p_3(G) < p_3(G')$, and $|p_0(G') - p_3(G')| \leq O(\frac{1}{n})$. This G' is also the disjoint union of vertex-disjoint cliques and has no more cliques than G .

To construct G' we sequentially move vertices from the smallest clique³ in G to the largest one, breaking ties arbitrarily, thereby changing p_0 and p_3 by at most $O(\frac{1}{n})$. We stop when $|p_0 - p_3| \leq O(\frac{1}{n})$.

The case $p_0(G) < p_3(G)$ is similar, but even simpler. We sequentially isolate vertices until $|p_0 - p_3| \leq O(\frac{1}{n})$.

The last reduction is achieved by passing to a subsequence in which the limits $p_0 = p_0(\mathcal{G})$ and $p_3 = p_3(\mathcal{G})$ exist and are equal.

By passing to a subsequence if necessary we can fix the bound r on the number of nontrivial cliques and the relative sizes $\alpha_1, \alpha_2, \dots, \alpha_r \geq 0$ of these cliques. In other words, we can now restrict ourselves to a sequence \mathcal{G} whose n th graph is $G_n = K_{\alpha_1 n} \sqcup K_{\alpha_2 n} \sqcup \dots \sqcup K_{\alpha_r n} \sqcup \bar{K}_{\beta n}$ where $\alpha_1, \alpha_2, \dots, \alpha_r \geq 0$ and $\beta = 1 - \sum \alpha_i \geq 0$. We ignore issues of rounding $\alpha_j n$ to integral values since this affects the relevant parameters by only an additive $O(\frac{1}{n})$ term⁴. The next lemma deals with graphs of this particular structure.

Lemma 2.1. *Let $\alpha_1, \dots, \alpha_r \geq 0$ and $\beta = 1 - \sum \alpha_i \geq 0$. Let*

$$p_3 = \sum \alpha_i^3 \quad \text{and} \quad p_0 = 6 \sum_{i < j < k} \alpha_i \alpha_j \alpha_k + 6\beta \sum_{i < j} \alpha_i \alpha_j + 3\beta^2 \sum \alpha_i + \beta^3.$$

If $p_0 = p_3$ then

$$p_0, p_3 \geq \rho = 0.159181 \dots$$

This bound is tight.

Proof. We apply the Lagrange multipliers method to determine the smallest possible value of $\max(p_0, p_3)$ under the constraints $p_0 = p_3, \alpha_i \geq 0, \sum \alpha_i \leq 1$. (We eliminate the variable β by substituting $\beta = 1 - \sum \alpha_i$). There are three cases to consider:

- The minimum is attained in the interior of this region. We calculate the partial derivatives of the objective function $\frac{\partial p_3}{\partial \alpha_i} = 3\alpha_i^2$ and the derivatives of the constraint

³If there is an isolated vertex in the graph, the corresponding clique gets eliminated, but this creates no problem in the argument.

⁴To see this designate one vertex in each nontrivial clique as "special". The difference between the calculations below and the exact values comes only from triples that contain a special vertex.

$$\begin{aligned} \frac{\partial(p_3 - p_0)}{\partial\alpha_l} &= 3\alpha_l^2 - 6 \sum_{i < j, i, j \neq l} \alpha_i \alpha_j - 6 \left(1 - \sum_i \alpha_i\right) \sum_{i \neq l} \alpha_i + 6 \sum_{i < j} \alpha_i \alpha_j \\ &\quad - 3 \left(1 - \sum_i \alpha_i\right)^2 + 6 \left(1 - \sum_i \alpha_i\right) \sum_i \alpha_i + 3 \left(1 - \sum_i \alpha_i\right)^2 \\ &= 3\alpha_l^2 + 6\alpha_l \sum_{i \neq l} \alpha_i + 6 \left(1 - \sum_i \alpha_i\right) \alpha_l = 3\alpha_l(\alpha_l + 2(1 - \alpha_l)). \end{aligned}$$

The Lagrange multipliers method implies that at a critical point there holds $\frac{\partial p_3}{\partial \alpha_l} = \lambda \frac{\partial(p_3 - p_0)}{\partial \alpha_l}$, where λ is a Lagrange multiplier. Consequently $\alpha_l = \lambda(2 - \alpha_l)$ for all l (since we are working in the interior of our region, all α_i are positive). This is a linear equation, so all α_l are equal. If $r \geq 3$, then $p_3 \leq 3(\frac{1}{3})^3 = \frac{1}{9}$, and $p_0 \geq \frac{1}{4} - p_3 > 0.13$ (by Goodman’s theorem), hence $p_0 \neq p_3$, a contradiction. If $r = 1$, $p_3 = \alpha_1^3 = p_0 = (1 - \alpha_1)^3 + 3(1 - \alpha_1)^2 \alpha_1$. The solution is $\alpha_1 = 0.652704\dots$, for which $p_0 = p_3 = 0.278\dots > \rho$. If $r = 2$, $p_3 = 2\alpha_1^3 = p_0 = (1 - 2\alpha_1)^3 + 3(1 - 2\alpha_1)^2 \cdot 2\alpha_1 + 6(1 - 2\alpha_1)\alpha_1^2$. The solution is $\alpha_1 = 0.442125\dots$, and $p_0 = p_3 = 0.172848\dots > \rho$.

- The minimum is attained when $\alpha_i = 0$ for some i . This case is solved by removing this α_i using induction on r .
- The minimum is attained when $\forall i \alpha_i > 0$, and $\sum \alpha_i = 1$. We add the constraint $\sum \alpha_i = 1$ to our Lagrange multipliers equations. This gives $\alpha_i^2 = \lambda(2\alpha_i - \alpha_i^2) + \mu$. All α_i satisfy this quadratic equation, so they all take at most two different values, say α_1 appears s times and α_2 appears t times with $s\alpha_1 + t\alpha_2 = 1$. We can assume that $s, t > 0$, and $\alpha_1 > \alpha_2 > 0$ since the case of equal α ’s was already treated above.

If $s \geq 3$ then $p_3 \leq \frac{1}{9}$, and it follows (as before), that $p_0 \neq p_3$, a contradiction.

If $s = 1$, $p_3 = \alpha_1^3 + t\alpha_2^3$, $p_0 = 6\binom{t}{2}\alpha_1\alpha_2^2 + 6\binom{t}{3}\alpha_2^3$, and $\alpha_1 = 1 - t\alpha_2$. Denote $x = t\alpha_2$. Here, $0 < x < \frac{t}{t+1}$. We have $p_3 = (1 - x)^3 + \frac{x^3}{t^2}$ and $p_0 = 3x^2 - 2x^3 - \frac{3}{t}x^2 + \frac{2}{t^2}x^3$. Let $\tau(x) = p_3 - p_0$. The value of x is determined by the equation $\tau(x) = 0$. Note that $\tau(0) > 0$, and $\tau'(x) < 0$ for $0 < x < \frac{t}{t+1}$. Therefore, τ is decreasing, and there is a unique solution for $\tau(x) = 0$.

Now, $\tau(\frac{1}{3}) = \frac{1}{27} + \frac{1}{3t} - \frac{1}{27t^2} > 0$ implies that $x > \frac{1}{3}$. This implies that $p_0(x) > p_0(\frac{1}{3})$, since, $p'_0 > 0$. It remains to compute, for $t \geq 4$, $p_0(\frac{1}{3}) > \frac{7}{27} - \frac{1}{3t} \geq \frac{7}{27} - \frac{1}{12} = 0.175\dots > \rho$.

The case $s = t = 1$ is vacuous, since $p_3 > p_0 = 0$.

If $s = 1, t = 2$ the equation in $x = 2\alpha_2$ is $1 - 3x + \frac{3}{2}x^2 + \frac{3}{4}x^3 = 0$ with root at $x = 0.469285\dots$, and $p_0 = p_3 = 0.1753\dots > \rho$.

If $s = 1, t = 3$ then $x = 3\alpha_2$ satisfies $1 - 3x + x^2 + \frac{8}{9}x^3 = 0$ so that $x = 0.409632\dots$, and $p_0 = p_3 = 0.2134\dots > \rho$.

This concludes the case $s = 1$, and the only remaining case to analyze is $s = 2$. Again, $x := t\alpha_2$. Here, $p_3 = \frac{(1-x)^3}{4} + \frac{x^3}{t^2}$ and $p_0 = \frac{3}{2}x - \frac{1}{2}x^3 - \frac{3}{t}x^2 + \frac{2}{t^2}x^3$. The range of x is $0 < x < \frac{t}{t+2}$. We first consider the case $t \geq 3$. Define $\tau(x) = p_3(x) - p_0(x) = \frac{1}{4} - \frac{9}{4}x + \frac{3}{4}x^2 + \frac{1}{4}x^3 + \frac{3}{t}x^2 - \frac{1}{t^2}x^3$. Again, $\tau(0) > 0$, and $\tau' < 0$ for $0 < x < \frac{t}{t+2}$, hence τ decreases. Also, p_0 increases, since $p'_0 > 0$.

Let $x_0 = 0.115749\dots$ be the solution in $[0, 1]$ of the equation $\frac{1}{4} - \frac{9}{4}x + \frac{3}{4}x^2 + \frac{1}{4}x^3 = 0$. $\tau(x_0) = \frac{3}{7}x_0^2 - \frac{1}{7^2}x_0^3 > 0$. Since τ decreases, the solution for $\tau = 0$ is bigger than x_0 . Since p_0 increases, the optimal value of p_0 is larger than $p_0(x_0) = 0.172848\dots - \frac{0.040193\dots}{t} + \frac{0.003102\dots}{t^2} \geq 0.159450\dots > \rho$.

If $s = 2, t = 1$, then $p_0 = 6\alpha_1^2(1 - 2\alpha_1)$, and $p_3 = 2\alpha_1^3 + (1 - 2\alpha_1)^3$. Solving for $p_0 = p_3$ gives $\alpha_1 = 0.234643\dots$ or $\alpha_1 = 0.427373\dots$. Since $\alpha_1 > \alpha_2 = 1 - 2\alpha_1$, we have $\alpha_1 = 0.427373\dots$, and $p_0 = p_3 = \rho$. This example proves the tightness claim in the lemma.

Finally, $s = 2, t = 2$ gives $p_0 = 6(2\alpha_1^2\alpha_2 + 2\alpha_1\alpha_2^2) = 3\alpha_1(1 - 2\alpha_1)$ and $p_3 = 2\alpha_1^3 + 2\alpha_2^3 = \frac{1}{4} - \frac{3}{2}\alpha_1 + 3\alpha_1^2$. Solving for $p_0 = p_3$ with $\alpha_1 > \alpha_2$ gives $\alpha_1 = \frac{3+\sqrt{5}}{12} = 0.436338\dots$, and $p_0 = p_3 = \frac{1}{6} > \rho$. ■

3. ON 4-PROFILES OF TOURNAMENTS

As in the discussion above, we consider families \mathcal{T} of tournaments of orders going to ∞ and discuss their k -local profiles. Likewise we define the limit values sets $\pi_l(\mathcal{T})$ and the limit sets Π_l of tournaments. The 3-profiles of tournaments are easy and completely understood. There are just two 3-vertex tournaments, one transitive and one cyclic with frequencies t_3 and $c_3 = 1 - t_3$, respectively. It is well-known and easy to prove that for every \mathcal{T} there holds $t_3(\mathcal{T}) \geq \frac{3}{4}$ and this is all there is to 3-profiles of tournaments.

In this section we prove the analog of Theorem 1.7 for tournaments and $k = 4$. In addition we derive some information on Π_4 (tournaments).

There are exactly four isomorphism types of 4-vertex tournaments, see Figure 1. Their names are as follows:

- T_4 is the transitive 4-tournament.
- C_4 is the (one and only) strongly connected 4-tournament.
- In W_4 there is a cyclic triangle all three vertices of which arrow the fourth vertex.
- In L_4 one vertex arrows all the three vertices of a cyclic triangle.

We use the shorthand $t_4(T), c_4(T), w_4(T), l_4(T)$ for $p(T_4, T)$, etc., or even do not mention T explicitly when clear from the context. Note that if the limits t_4, c_4, w_4, l_4 exist for some family of tournaments \mathcal{T} , then the limit fraction of cyclic triangles $c_3(\mathcal{T})$ exists as well and equals $\frac{2c_4+w_4+l_4}{4}$.

We recall the class $\mathcal{C} = C_n$ of *circular tournaments* of odd order n . The n vertices of C_n are equally spaced on the unit circle, with an edge $x \rightarrow y$ iff the clockwise arc from x to y is shorter than the counter clockwise arc between them. We are now ready to state our theorem for tournaments:

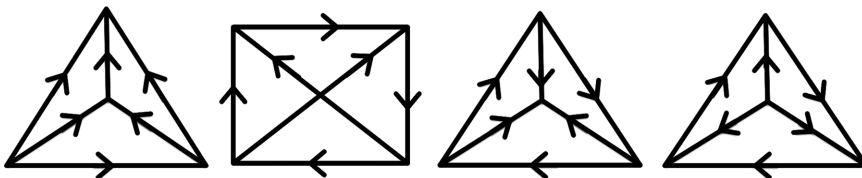


FIGURE 1. The four types of 4-vertex tournaments (in order): T_4, C_4, W_4, L_4 .

Theorem 3.1. Every family of tournaments \mathcal{T} for which $\overline{t}_4(\mathcal{T}) < \frac{1}{2}$ is 4-universal. Moreover, $\underline{w}_4(\mathcal{T}), \underline{l}_4(\mathcal{T}) \geq \frac{1}{2} - \overline{t}_4(\mathcal{T})$. Also $\underline{c}_4 \geq \frac{1}{6}$ when $\overline{t}_4(\mathcal{T}) \leq \frac{1}{2}$.

The circular tournaments satisfy $t_4(\mathcal{C}) = \frac{1}{2}$ and yet $l_4 = w_4 = 0$.

Remark 3.2. We do not know whether the inequality $\underline{c}_4 \geq \frac{1}{6}$ is tight, and so we ask how small $c_4(\mathcal{T})$ can be when $t_4(\mathcal{T}) = \frac{1}{2}$. A similar question is presented in remark 3.5.

Proof. The theorem follows from the proposition below. In more detail, the positivity of t_4, l_4 , and w_4 follows from items (2), (3), and (4), respectively. The lower bound on c_4 follows by combining 1 with the equality $c_3 = \frac{1-t_4+c_4}{4}$. ■

Proposition 3.3. The following inequalities hold for every $(t_4, c_4, w_4, l_4) \in \Pi_4$.

$$c_4 \leq t_4 \tag{1}$$

$$\frac{3}{8} \leq t_4 \tag{2}$$

$$t_4 + l_4 \geq \frac{1}{2} \tag{3}$$

$$t_4 + w_4 \geq \frac{1}{2} \tag{4}$$

All the above inequalities are tight. In addition

$$c_4 \geq 6c_3^2. \tag{5}$$

Remark 3.4. These inequalities, four linear and one quadratic, provide some information on the set Π_4 . It would be interesting to derive a full description of Π_4 .

Remark 3.5. We still do not know how tight inequality (5) is and we ask how small $c_4(\mathcal{T})$ can be, given $c_3(\mathcal{T})$. It is not difficult to see that this question is equivalent to the problem of minimizing $t_4(\mathcal{T})$ given $t_3(\mathcal{T})$, which is analogous to an interesting question about graphs: Let $2 \leq s < r$, given $p(K_s, \mathcal{G})$ how small can $p(K_r, \mathcal{G})$ be? (This question is stated in its general form in [11] though it was probably posed earlier.) Razborov’s recent solution for $s = 2, r = 3$ [16] was a major achievement in local graph theory. The problem was subsequently solved for $s = 2, r = 4$ by Nikiforov [13], and for $s = 2$, and general r by Reiher [17]. To the best of our knowledge, the problem remains open for $s \geq 3$.

Proof of Proposition 3.3. Inequality (1): Recall that $t_3 \geq \frac{3}{4}$ and $c_3 \leq \frac{1}{4}$. The inequality follows, since, $c_3 = \frac{2c_4+l_4+w_4}{4}$. This holds with equality for the circular tournaments \mathcal{C} , for which $t_4 = c_4 = \frac{1}{2}$ and $l_4 = w_4 = 0$.

Inequality (2) follows by applying the inequality $t_3 \geq \frac{3}{4}$ to the outset of every vertex. To see that there are always at least $\frac{3}{4} \sum \binom{d^+(x)}{3}$ transitive 4-vertex subtournaments, count for each vertex x , the number of transitive triangles among the $d^+(x)$ out-neighbors of x . The inequality follows now from the convexity of the function $\binom{t}{3}$. Equality holds, e.g., for random tournaments.

Inequalities (3) and (4) are equivalent, of course. We prove the latter. Clearly $\binom{n}{4}(t_4 + w_4) = \sum_{x \in V} \binom{d^+(x)}{3}$. Again a simple convexity argument yields the inequality and equality holds for random tournaments.

Inequality (5): Note that two cyclic triangles sharing a common edge necessarily form a C_4 . Let us denote by S the set of cyclic triangles in an n -vertex tournament T , and for an

edge e , $S_e = \{s \in S : e \subset s\}$. Then $c_4 \binom{n}{4} = \sum_e \binom{|S_e|}{2} \geq \binom{n}{2} \left(\frac{\sum |S_e|}{2} \binom{n}{2} \right) = \binom{n}{2} \left(\frac{3c_3 \binom{n}{3}}{2} \binom{n}{2} \right) = (6 + o(1))c_3^2 \binom{n}{4}$. The inequality follows. ■

4. FURTHER DIRECTIONS AND DISCUSSION

The following questions suggest themselves:

- (1) Is there some $\epsilon > 0$ such that every graph family with $p_0, p_3 < \frac{1}{8} + \epsilon$ is 4-universal? As observed by Mykhaylo Tyomkyn (personal communication), no such condition yields 5-universality, see below.
- (2) Is there some $\epsilon > 0$ such that every graph family \mathcal{G} with $p(K_4, \mathcal{G}), p(\overline{K}_4, \mathcal{G}) < \frac{1}{64} + \epsilon$ is l -universal for some values of $l \geq 3$?
- (3) What are the triples k, l, r for which there exists an $\epsilon > 0$ such that the conditions $p(K_k, \mathcal{G}) < 2^{-\binom{k}{2}} + \epsilon$ and $p(\overline{K}_r, \mathcal{G}) < 2^{-\binom{r}{2}} + \epsilon$ imply l -universality?
- (4) Is there some $\epsilon > 0$ such that every tournaments family with $t_4 < \frac{3}{8} + \epsilon$ is necessarily 5-universal? What about l -universality for bigger l ?
- (5) Does $t_5 < \frac{51}{2^{10}} + \epsilon$ imply l -universality for some values of $l \geq 4$?
- (6) For which integers k, l does there exist an $\epsilon > 0$ such that every tournament satisfying $t_k < \frac{k!}{2^{\binom{k}{2}}} + \epsilon$ is l -universal? (Here t_k is the proportion of transitive k -vertex subtournaments).
- (7) Jacob Fox has raised the question whether problem 1.4 can have a positive answer only with $l = O(k)$. As he pointed out, this would follow from the existence of a large k -clique-free graph G with $p(\overline{K}_k, G) < 2^{-\binom{k}{2}} + o_{|G|}(1)$. We note that it is an old and intriguing problem how small $p(\overline{K}_k, G)$ can be for a large k -clique-free graph. See [4] and the recent work [14].
- (8) What are the possible values of $t_k(\mathcal{T})$, given the value of $t_l(\mathcal{T})$? Here \mathcal{T} is a family of tournaments and $k > l \geq 3$ are integers. The first interesting (and open) case is $k = 4, l = 3$.
- (9) In this article we discuss how the paucity of small homogeneous sets implies universality in graphs and in tournaments. It is conceivable that these two problem sets can be connected, perhaps in the spirit of Alon et al. [2], but we do not know how or whether this can be done. Specifically, can some connection can be established between items 3 (say, with $k = r$) and 6 above?

We now present Jacob Fox's proof of Proposition 1.6 which uses the following result of Prömel and Rödl. For a simpler proof of the results from [15] with improved constants see [6].

Proposition 4.1. *For every $a > 0$ there is a $b > 0$ such that every n -vertex graph G with $\alpha(G), \omega(G) < a \log n$ is $b \log n$ -universal.*

Proof of Proposition 1.6. Let $G \in \mathcal{G}$, be an n -vertex graph. Select H as a random subgraph of G with $m = 2^{k/4}$ vertices. The expected number of k -cliques and k -anticliques in H is at most $\binom{m}{k} (p(K_k, G) + p(\overline{K}_k, G)) \leq m^k \cdot 2 \cdot (2^{-\binom{k}{2}} + \epsilon + o_n(1))$ which can be made smaller than 1 by making k large enough. Therefore, such an H exists with $\alpha(H), \omega(H) < k$. Proposition 4.1 implies that H is ck -universal, where c is a constant. This clearly implies that G is ck -universal, as claimed. ■

Two comments are in order here: (i) The above argument applies, with some minor modifications to item 3 above and yields that if r and k are of the same order, then l could be of the same order as well. (ii) Jacob Fox has pointed out that the methods of [6] can be adapted to yield an analogue of the simple proof for Proposition 4.1 for tournaments, namely

Proposition 4.2. *For every $a > 0$ there is a $b > 0$ such that every n -vertex tournament T with $tr(T) < a \log n$ is $b \log n$ -universal.*

From this we can easily deduce

Lemma 4.3. *Every family of tournaments \mathcal{T} for which $t_k(\mathcal{T}) < k!2^{-\binom{k}{2}} + \epsilon$ is ck -universal for some absolute $c > 0$.*

Proof. Apply the above proof of Proposition 1.6 to an arbitrary large $T \in \mathcal{T}$. It shows that a random subset S of $2^{k/4}$ vertices in T contains no transitive k -vertex subtournament. By Proposition 4.2 the tournament induced on S , and therefore the whole of T is ck -universal. ■

Mykhaylo Tyomkyn (personal communication) found the following recursive construction which is not 5-universal even though $p_0, p_3 \leq \frac{1}{8}$. Let $G_1 = C_5$ be the pentagon graph. To construct G_n , take 5 blocks each being a copy of G_{n-1} and connect every two consecutive (modulo 5) copies by a complete bipartite graph. It is easy to see that the graph G_n is self-complementary, and $p_3(G_n) = \frac{1}{25}p_3(G_{n-1}) + \frac{6}{25}e(G_{n-1})$ where $e(G_{n-1}) = \frac{1}{2}$ is the edge density of G_{n-1} . It follows that $\lim_n p_3(G_n) = \frac{1}{8} = \lim_n p_0(G_n)$. On the other hand, this family of graphs is not 5-universal. We prove by induction that G_n has no induced copy of a 5-vertex path x_1, \dots, x_5 . By induction not all 5 vertices are in the same block. Also, if they all reside in different blocks then x_1 and x_5 are adjacent which is impossible. So let x_u, x_v with $u < v$ reside in the same block and x_w be in a neighboring block. Then necessarily $u = w - 1, v = w + 1$. But at least one of x_u, x_v has a neighbor other than x_w and this vertex cannot be fit into any of the blocks.

Notes added in proof:

- (1) In a follow up paper, Hefetz and Tyomkin [10] settle Problem 1 in the above list and make several additional interesting contributions to this area.
- (2) We have recently made some progress on Problem 8 above. We intend to publish our results soon.

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