Truth Revelation in Rapid, Approximately Efficient Combinatorial Auctions

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Abstract

Some important classical mechanisms considered in Microeconomics and Game Theory require the solution of a difficult optimization problem. This is true of mechanisms for combinatorial auctions, which have in recent years assumed practical importance, and in particular of the gold standard for combinatorial auctions, the Generalized Vickrey Auction (GVA). Traditional analysis of these mechanisms - in particular, their truth revelation properties - assumes that the optimization problems are solved precisely. In reality, these optimization problems can usually be solved only in an approximate fashion. We investigate the impact on such mechanisms of replacing exact solutions by approximate ones. Specifically, we look at a particular greedy optimization method, which has empirically been shown to perform well. We show that the GVA payment scheme does not provide for a truth revealing mechanism. We introduce another scheme that does guarantee truthfulness for a restricted class of players. We demonstrate the latter property by identifying sufficient conditions for a combinatorial auction to be truth-revealing, conditions which have applicability beyond the specific auction studied here.

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1 Introduction

This articles concerns combinatorial auctions (also called combinational), that is, auctions in which multiple goods are available and in which bidders can post bids for subsets, i.e. bundles, of the goods. Such auctions have become the object of increased interest recently, in part because of the general interest in auctions, and in part because of specific auctions in which combinatorial bidding would seem natural, such as the series of the FCC spectrum auctions [7, 2, 8]. [1]

Combinatorial auctions (henceforth CAs) typically require the solution of one or more difficult optimization problems. The computational complexity of these problems threatens to render the traditional auction designs a mere theoretical construct. One approach to meeting this threat is to replace the exact optimization by an approximate one. This, however, gives rise to a new challenge: traditional analysis of established CA mechanisms relies strongly on the fact that the goods are allocated in an optimal manner, and the properties guaranteed by the mechanism (such as truthful bidding, to be defined later), disappear if the allocation is anything less than optimal. This is true in particular of the Generalized Vickrey Auction (GVA), also defined later, which is widely taken to be the gold standard for CAs. The primary focus of this article is to present a simple approximate optimization method for CAs that possesses two attractive properties:

- the method performs a reasonably effective optimization, and
- there exists a novel payment scheme which, when coupled with the approximate optimization method, makes for a combinatorial auction in which truth-telling is a dominant strategy.

In order to show the latter property we identify several axioms which are sufficient to ensure truth-telling in any combinatorial auction; these axioms are interesting in there own right, as they can be applied to auctions other than the one discussed here. *Note*: Since we aim at making this article easily accessible to both computer scientists and game theorists, we include some rather basic material.

[1] Up until now the FCC auctions have not in fact been combinatorial, due in part to the complexity problem discussed below. However, the Federal Energy Regulatory Commission has mandated that the next auction be combinatorial.

2 Mechanism design for CA

In this section, we consider the design of combinatorial auctions as a problem of designing a game of incomplete information for which the weakly-dominant strategies present a *good* way of allocating the goods and paying for them. The general setting is that of economic mechanism design: see [6, Chapter 23], for example, for an introduction to the field and [12] for a description of auctions in this framework. Contrary to the latter, we shall restrict our description to combinatorial auctions in which no externalities can be expressed. Informally, each bidder sends a message describing (truthfully or not) his/her preferences, the auctioneer, then, computes the resulting allocation of the goods and the payments, based on the bidders' messages but according to rules known in advance. The mechanism is a *truthful* one if it is in the best interest of the bidders to send messages that truthfully reveal their preferences.

Formally, we consider a set P of n bidders. A bidder knows his own utility function, i.e., his *type*, but this information is private and neither the auctioneer nor the other players have access to it. The final result of an auction consists of two elements: an allocation of the goods and a vector of payments from the bidders to the auctioneer, both of which are functions of the bidders' signals, i.e. messages, or declarations. We have a finite set G of k goods and an allocation is a *partial* function from G to P, i.e. a function $a : G \rightarrow P'$, with $P' = P \cup \{unallocated\}$, since we do not insist that all goods be allocated. The set of outcomes, i.e. allocations, is $\mathcal{O} = P'^G$, the set of partial functions from G to P. Since we do not allow for externalities, the set Θ_i of the possible types for bidder i is ${\mathbf{R}_+}^{2^G}$, where $\mathbf{R}_+$ is the set of all nonnegative real numbers. Since the set Θ_i does not depend on i, we shall write Θ. An element of Θ assigns a real nonnegative valuation to every possible bundle. The set Θ is also the set of messages that bidder i may send. A bidder may send any element of Θ, irrespective of his (true) type, i.e. a bidder may lie. We shall typically use t to denote a (true) type, d to denote a message, T or D to denote vectors of n types and P for a payment vector, i.e. a vector of n nonnegative numbers.

Since we assume the Independent Value Model and Quasi-Linear utilities, fairly standard assumptions in the field, the utility, for a bidder of type t, of bundle s and payment x is:

$$u = t(s) - x \tag{1}$$

Definition 1 *A (direct) mechanism for combinatorial auctions consists of*

- *an allocation algorithm f that picks, for each vector D (D is a vector of declared types), an allocation $f(D)$,*
- *a payment scheme p that determines, for each vector D a payment vector $p(D)$: $p_i(D)$ is paid by bidder i to the auctioneer.*

Let us denote the bundle obtained by i as:

$$g_i(D) = f(D)^{-1}(i) \tag{2}$$

Notation: In general g_i depends on the allocation algorithm f, but when f is clear from the context we shall abuse the notation and treat g_i as a direct function of the bid vector, D. Equation 1 implies that if bidder i has (true) type t, his utility from the mechanism is:

$$u_i = t(g_i(D)) - p_i(D), \tag{3}$$

where $D = \langle d_1, \ldots, d_n \rangle$ is the vector of declarations.

The first term of this sum, $t(g_i(D))$ is often called the valuation of i: $v_i(f(D), t)$. The game-theoretic solution concept used throughout this paper is that of a weakly-dominant strategy, that is a strategy that is as good as any other for a given player, no matter what other players do. This is in contrast with the weaker and more common notion of Nash equilibria. The particular property we would like to ensure for our mechanism is that the dominant strategy for each player is to bid his true valuation; in other words, no bidder can be better-off by lying, no matter how other bidders behave. This is obviously a very strong requirement.

Definition 2 *A mechanism $\langle f, p \rangle$ is truthful iff for every $i \in P$, $t \in \Theta$ and any vector D of declarations, if D' is the vector obtained from D by replacing the i-th coordinate d_i by t, then: $t(g_i(D')) - p_i(D') \geq t(g_i D)) - p_i(D)$.*

3 The generalized Vickrey auction

A very general method for design truthful mechanisms has been devised by Clarke and Groves [1, 4]. Applied to combinatorial auctions it generalizes the second price auctions of Vickrey [13]. We shall now describe those generalized

Vickrey auctions, prove that the mechanism described is truthful and then discuss the complexity issues that render those auctions infeasible when k, the number of goods, is large. Generalized Vickrey Auctions (GVA's) appear to be part of the folklore of mechanism design. A general description of the GVA may be found in [5, 12]; we adopt a special case of it, one which does not allow for externalities.

Given a vector D of declarations, the generalized Vickrey auction defines the allocation and payment policies as follows (notice that $a^{-1}(i)$ is the bundle allocated to i by allocation a, and that g_i is defined in Equation 2):

$$f(D) = \mathrm{argmax}_{a \in \mathcal{O}} \sum_{i=1}^{n} d_i(a^{-1}(i)), \tag{4}$$

$$p_j(D) = - \sum_{i=1, i \neq j}^{n} d_i(g_i(D)) + \sum_{i=1, i \neq j}^{n} d_i(g_i(Z)) \tag{5}$$

where $Z_i = D_i$ for any $i \neq j$ and $Z_j(s) = 0$ for any bundle $s \subseteq G$. A proof of the truthfulness of the Clarke-Groves-Vickrey mechanism may be found, for example in [6, Proposition 23.C.4].

Since bidders truthfully declare their type and the allocation maximizes the sum of the declared utilities, a GVA's allocation maximizes the sum of the true valuations of the bidders, i.e. the social welfare. In a quasi-linear setting, this is equivalent to Pareto optimality. Therefore a GVA is Pareto optimal. The mechanism to be presented in section 8 only approximately maximizes the sum of the true valuations of the bidders, and is not Pareto optimal.

As we discuss in the following sections, it is known that algorithmic complexity considerations imply that Pareto efficiency cannot be feasibly attained. Specifically, ensuring Pareto efficiency requires solving an intractable optimization problem. This is true even if we restrict the class of bidders severely, as we propose in section 4.

4 Single-minded bidders

As is customary, we shall consider that any algorithm whose running-time is polynomial in k and n is feasible, but any algorithm whose running-time is not polynomial in k or in n is infeasible. The size of the set $\mathcal{O}$ of allocations is exponential in k, if there are at least two bidders, and the set Θ of possible types is doubly-exponential in k. Since, in a direct mechanism (we consider no others), the message that a bidder sends describes one specific element (type) of Θ, a bidder needs an exponential number of bits to describe his type: the length of the messages sent in any such mechanism, and in a generalized Vickrey auction, is exponential in k, therefore infeasible. The design of a feasible version of the GVA must begin, therefore, by reducing the set of possible types to some set of singly-exponential size. All implementations of auctions assume that the bidders express their preferences by a small set of *bids*. We shall start with a most sweeping restriction: in Section 9 we shall consider relaxing this restriction.

We shall assume that bidders are single-minded and care only about one specific (bidder-dependent) set of goods: if they do not get this set they value the outcome at the lowest possible value: 0. In other words, our bidders are restricted to one single bid.

Definition 3 *Bidder i is* single-minded *iff there is a set $s \subset G$ of goods and a value $v \in \mathbf{R}_+$ such that its type t can be described as: $t(s') = v$ if $s \subseteq s'$ and $t(s') = 0$ otherwise.*

We shall denote by $\langle s, v \rangle$ the type just described. We shall assume, in most of this paper, that all bidders are single-minded, i.e. there are sets of goods s_i and nonnegative real numbers v_i such that bidder i is of type $\langle s_i, v_i \rangle$. We shall denote by Σ the set of all single-minded types. The size of the set Σ, contrary to the size of Θ, is singly-exponential in k. A string of polynomial size will be enough to code the declarations of the bidders: it will describe a set of goods and a value. In this setting, we identify bids and bidders.

We shall design a feasible truthful mechanism for combinatorial auctions among single-minded bidders. It may at first seem that this is a futile exercise, but at least anecdotal evidence suggests that this single-mindedness is not an uncommon situation. Indeed, R. Wilson [14] reports that, in the GVA used for selling timber harvesting rights in New Zealand, the bidders were almost single-minded: they were typically interested in all of the locations in a specific geographical area. It might also seem that this restriction does away with the computational issue; however, as we see in the next section, GVA is infeasible even with the restriction to single-minded bidders.

5 Infeasibility of the GVA

Let us now assume that all bidders are single-minded, i.e. the set of all possible types is now Σ.

In a GVA the allocation is the one defined in Equation 4. Computing this allocation requires optimizing $\sum_{i=1}^{n} d_i(a)$ over all a's in the set $\mathcal{O}$ that is of exponential size. Indeed, the problem of finding the allocation of Equation 4 has been shown in [10] to be infeasible. We remark that the restriction to single-minded bidders does nothing to alleviate the problem. Some care is needed in describing the NP-hardness result because we have two parameters to deal with: k, the number of goods and n, the number of bidders.

Theorem 1 *Let a single-minded type $d_i = \langle s_i, v_i \rangle$, $s_i \subseteq G$, $v_i \in \mathbf{R}_+$ be given for each bidder $i \in P$. Let $\mid G \mid = k$ and $\mid P \mid = n$. If k and n grow in a polynomially related way, the problem of finding an allocation a that maximizes $\sum_{i=1}^{n} d_i(a)$ is NP-hard. Moreover, the existence of a polynomial time algorithm guaranteed to find an allocation whose value is at least $k^{-1/2+\epsilon}$ times the value of the optimal solution would have sweeping theoretical consequences (NP = co-RP).*

In Theorem 1, we required k and n to grow in a polynomially related way. This restriction is needed. On one hand, if $n \leq \log k$, an optimal allocation may be found in time linear in k. On the other hand, if $k \leq \log n$, then dynamic programming provides an optimal allocation in time quadratic in n, as shown in [10].

Let us now consider the significance of Theorem 1. Even if (single-minded) bidders declare truthfully their type, we cannot always attain an efficient allocation. Global restrictions on the structure of the set of bidders are considered in [10] and shown to allow a polynomial search for the efficient allocation. They restrict the possible types of the bidders to a small subset of Σ, based on some inherent structure of G. It does not seem those restrictions can be met in practice.

If the number of goods is large, we may either find an algorithm that computes the efficient allocation but may, in the worst cases, never terminate (for all practical purposes) or settle for an algorithm that provides a sub-efficient allocation. Both ideas have been proposed in [3, 11]. But the impact of such an approximation on the quality of the mechanism, i.e. its truthfulness, or the revenue it generates, has not been studied. A pioneering study of the properties of approximate mechanisms, but not for combinatorial auctions, may be found in [9]. In section 6, we shall provide a feasible approximation algorithm that appears to be very effective in practice and, in section 8, we shall describe a payment scheme, different from the GVA scheme, that guarantees truthfulness. The payment scheme is carefully tailored to the specific approximation algorithm.

6 The greedy allocation

Since an efficient solution seems out of reach, we shall look for an approximately efficient solution. We shall propose a family of algorithms that provide such an approximation. Each of those algorithms runs in almost linear time in n, the number of single-minded bidders. One algorithm of the family guarantees an approximation ratio of $k^{-1/2}$.

A single-minded bidder declaring $\langle s, a \rangle$, with $s \subseteq G$ and $a \in \mathbf{R}_+$ will be said to put out a bid $b = \langle s, a \rangle$. We shall use $s(b)$ and $a(b)$ to denote the components of b and call $a(b)$ the *amount* of the bid b. As explained in section 4, we identify bids and bidders. Two bids $b = \langle s, a \rangle$ and $b' = \langle s', a' \rangle$ conflict if $s \cap s' \neq \emptyset$.

The algorithms we consider execute in two phases.

- in the first phase, the bids are sorted by some criterion. The algorithms of the family are distinguished by the different criteria they use. Since there are n bids, this phase takes time of the order of $n \log n$. We assume a criterion, i.e. a norm is defined and the bids are sorted in decreasing order following this norm.

- in the second phase, a greedy algorithm generates an allocation. Let L be the list of sorted bids obtained in the first phase. The first bid of L, say $b = \langle s, a \rangle$ is granted, i.e. the set s will be allocated to b and then the algorithm examines each bid of L, in order, and grants it if it does not conflict with any of the bids previously granted. If it does, it denies, i.e. does not grant, the bid. This phase requires time linear in n.

The use of such a greedy scheme is very straightforward and speedy. We shall now discuss its efficiency: how efficient is the allocation generated? The efficiency of the allocation generated depends obviously both on the criterion used in the first phase and on the types of the bidders, or on the distribution with which the bidders are generated. It is clear that, to obtain allocations close to efficiency, one should use a norm that pushes bids that have a good chance to be part of an efficient allocation towards the beginning of the list L. The amount of a bid is a good criterion in this respect: we want bids with higher amounts to have a larger norm than bids with lower amounts, at least when the bids are for the same set of goods. Similarly, leaving the amount of a bid unchanged but making its bundle a smaller set (inclusion-wise), should also increase the norm. We shall require that changing s to s' with $s' \subset s$ or changing v to v' with $v' > v$ increase the norm of a bid. Let us call this property *bid-monotonicity*. This is the only requirement we shall make. Many criteria satisfy it.

In real-life situations, one can, typically, find a suitable natural norm related to the economic parameters of the bundle that measures the a-priori attractiveness of the bid (for the auctioneer). In the FCC auction, goods (licenses) are characterized by the population they cover. The (inverse of the) sum of those populations is a good indicator. In the abstract, if we know nothing concrete about the goods, our best bet is to use the size of the set of goods mentioned in a bid. We shall look in particular at the average-amount-per-good measure.

Definition 4 *The average amount per good of a bid* $b = \langle s, a \rangle$ *is* $\frac{a}{|s|}$.

Sorting the list L by descending average amount per good is a very reasonable idea. But many other possibilities may be considered. Sorting L by descending amounts for example, or, more generally sorting L by a criterion of the form $\frac{a}{|s|^l}$ for some number l, $l \geq 0$, possibly depending on k. All such criteria satisfy bid-monotonicity.

How good is the greedy allocation in comparison with the optimal one? For $l = 1$, the worst case may be analyzed without much difficulty. The ratio between the total value of the allocation found by the greedy algorithm and that of the optimal allocation cannot be smaller than $1/k$, and this bound is tight. As usual in this sort of situations, on the average, on realistic distributions of bids, the performance of the greedy allocation scheme is much better than the lower bound above. We have been able to perform a full analysis of the worst case performance of those norms for different l's and found out that $l = 1/2$ is best: it guarantees an approximation ratio of at least $1/\sqrt{k}$ and, by Theorem 1, this is essentially the best one can hope for. The proof of the following is outside the bounds of this paper.

Proposition 1 *The greedy allocation scheme with norm $\frac{a}{|s|^{1/2}}$ approximates the optimal allocation within a factor of $1/\sqrt{k}$ and, unless NP = co-RP, one cannot achieve a substantially better ratio.*

Example 1 *Assume there are two goods a and b and three bidders Red, Green and Blue. Red bids 10 for a, Green bids 19 for the set $\{a, b\}$ and Blue bids 8 for b. We sort the bids by decreasing average amount and obtain: Red's bid for a (average 10), Green's bid for $\{a, b\}$ (average 9.5) and Blue's bid for b (average 8). The greedy algorithm grants Red's bid for a, denies Green's bid for $\{a, b\}$ since it conflicts with Red's and grants Blue's bid for b. The allocation is not efficient. The efficient allocation grants Green's bid for $\{a, b\}$ and denies both other bids.*

Our goal is to devise truthful mechanisms for combinatorial auctions among single-minded bidders. Given a suitable greedy allocation, can one find a payment scheme that makes the pair a truthful mechanism? A simple example suffices to show that Clarke's payment scheme is not suitable.

Example 2 *As in Example 1, there are two goods a and b and three bidders Red, Green, and Blue. Red bids 10 for a, Green bids 19 for the set $\{a, b\}$ and Blue bids 8 for b. The greedy algorithm grants Red's and Blue's bids and denies Green's bid, i.e. $f(D)(a) = Red$ and $f(D)(b) = Blue$. We shall now compute Red's payment. For this allocation we have the following declared valuations: $v_{Blue} = 8$ and $v_{Green} = 0$. If Red had bid zero, the greedy algorithm would have granted Green's bid and denied Blue's bid. Therefore, the allocation $f(Z)$ is defined by: $f(Z)(a) = f(Z)(b) = Green$, where $v_{Blue} = 0$ and $v_{Green} = 19$. Clarke's payment scheme gives to Red: $8 - 0$ for Blue and $0 - 19$ for Green, i.e., Red pays 11. Red ends up paying more than the amount he declared. If Red has been truthful and his valuation is indeed 10, his utility is -1. He would have been better off lying, under-bidding at, say 9, or 0. In such a case the greedy algorithm would have granted Green's bid and denied Blue's and Red's bids and the payment to Red would have been zero, making his utility 0, better than -1.*

7 A sufficient condition for a truthful mechanism

We shall describe in this section a number of properties of allocation schemes and of payment schemes for combinatorial auctions. Those properties seem natural properties to expect from a truthful mechanism and they are satisfied by the GVA. We shall then show that any mechanism that satisfies those properties is truthful. In section 8, we shall describe a payment scheme and show that the greedy allocation scheme, together with this new payment scheme, satisfy those properties.

The general structure of the properties of interest is that they consider a given set of single-minded types and vary *one* of those types. They restrict the changes that can appear in the allocation or the payments as a result of such a change. Let declarations be fixed, but arbitrary, for all bidders except j. Consider two possible declarations for j: $\langle s, v \rangle$ and $\langle s', v' \rangle$. Given an allocation scheme f and a payment scheme p, we shall consider the allocations and payments generated by both declarations of j. Let g_i be the set of goods obtained by bidder i if j declares $\langle s, v \rangle$, and g'_i the set he obtains if j declares $\langle s', v' \rangle$. Similarly denote by p_i and p'_i the payments of i.

Our first property requires that the allocation, among single-minded bidders, be exact, i.e. a single-minded bidder either gets exactly the set of goods he desires, nothing added, or he gets nothing.

$$\textbf{Exactness} \qquad \text{Either } g_j = s \text{ or } g_j = \emptyset$$

In an exact allocation, we shall say that j's bid is *granted* in the first case, and *denied* in the second case. In such a scheme, the allocation may be viewed as a set of bids (or bidders) that is conflict-free, i.e. the s coordinates have pairwise empty intersections.

Our next property, Allocation-Monotonicity, also concerns only the allocation scheme. It requires that, if j's bid is granted if he declares $\langle s, v \rangle$, it is also granted if he declares $\langle s', v' \rangle$ for any $s' \subseteq s$, $v' \geq v$. In other words: proposing more money for fewer goods cannot cause a bidder to loose his bid. It follows that, similarly, offering less money for more goods cannot cause a lost bid to win. Formally:

$$\textbf{A} - \textbf{Monotonicity} \quad s \subseteq g_j, \ s' \subseteq s, \ v' \geq v \ \Rightarrow \ s' \subseteq g'_j$$

We must immediately consider the consequences of A-Monotonicity, since we shall need them in stating the upcoming Critical property.

Lemma 1 *In a mechanism that satisfies Exactness and A-Monotonicity, given a bidder j, a set s of goods and declarations for all other bidders, there exists a critical value v_c such that*

$$\forall v, v < v_c \;\Rightarrow\; g_j = \emptyset,$$
$$\forall v, v > v_c \;\Rightarrow\; g_j = s,$$

We allow v_c to be infinite if $f(A^{s,v})^{-1}(j) = \emptyset$ for every v. Note that we do not know whether j's bid is granted or not in case $v = v_c$.

Our third property deals with a satisfied bidder: a satisfied bidder pays exactly the critical value of Lemma 1, i.e. the lowest value he could have declared and still be allocated the goods he desires.

$$\textbf{Critical} \qquad s \subseteq g_j \;\Rightarrow\; p_j = v_c$$

Notice that Critical says, first, that the payment for a bid that is granted does not depend on the amount of the bid, it depends only on the other bids. Then it says that it is exactly equal to the critical value below which the bid would have lost.

The fourth property concerns the payment scheme's reaction to a change in the set s requested while the amount v of the bid stays fixed. It requires that if you request more and get it, you never pay less. The idea is "you pay for what you get".

$$\textbf{P} - \textbf{Monotonicity} \qquad s \subseteq s' \subseteq g'_j,\; v = v' \;\Rightarrow\; p'_j \geq p_j$$

Notice that, under the assumptions, A-Monotonicity implies that $s \subseteq g_j$.

Our last property also concern the payment scheme. Together with Critical, it implies that the utility of no truthful bidder is negative. It concerns unsatisfied bidders, i.e., bids that are denied. We require that an unsatisfied bidder pay zero. The utility of an unsatisfied bidder is then zero. This is simply tuning the utility scales of the different bidders.

$$\textbf{Participation} \qquad s \not\subseteq g_j \;\Rightarrow\; p_j = 0$$

Any mechanism that satisfies the conditions above is truthful.

Theorem 2 *If a mechanism satisfies Exactness, A-Monotonicity, P-Monotonicity, Participation and Critical, then it is a truthful mechanism.*

8 A truthful mechanism with greedy allocation

We shall now describe the payment mechanism that we propose to be used in conjunction with the greedy allocation of section 6. The description of the payments is tightly linked with that of the greedy algorithm. The computation of the payment is performed in parallel with the execution of the greedy algorithm and takes time linear in the number of bidders for each payment. On the whole computing the allocation and the payments takes time at most quadratic in the number of bids.

We assume that the criterion used is average amount per good, the adaptation to most other suitable greedy allocations is obvious. Informally, a bidder pays, per good, the average price proposed by the first bid in the list L that is denied because of this bid. Consider a bid j in L. Let $c(j)$ be the average amount per good of j. We shall denote by $n(j)$ the first bid following j (bids are sorted in decreasing order, i.e. $c(j) \geq c(n(j))$) that has been denied but would have been granted were it not for the presence of j. Assume that such a bid exists. Notice that such a bid necessarily conflicts with j, and therefore:

$$n(j) = \min\{i \mid j < i, s(j) \cap s(i) \neq \emptyset, \forall l, l < i, l \neq j, l \text{ granted } \Rightarrow s(l) \cap s(i) = \emptyset\}.$$

Definition 5 (Greedy Payment Scheme) *Let L be the sorted list obtained in the first phase.*

- *j pays zero if his bid is denied or if there is no bid $n(j)$,*
- *if there is an $n(j)$ and j's bid $\langle s, v\rangle$ is granted he pays $\mid s \mid \times c(n(j))$.*

We may now state the main result of this paper.

Theorem 3 *The mechanism composed of the greedy allocation and payment schemes is truthful for single-minded bidders.*

Let us now describe this payment scheme on an example.

Example 3 *Consider the bidders of Example 2. The goods are a and b and the bidders are Red, Green, and Blue. Red bids 10 for a, Green bids 19 for the set $\{a, b\}$ and Blue bids 8 for b. We have seen that Red's and Blue's bids are granted, Green's bid is denied. This is not the efficient solution. If*

Red had not participated, Green's bid would have been the one with highest average price and would have been granted. Red pays Green's average price. Red pays 9.5*. Green pays* 0*, since his bid is denied. Blue pays* 0 *since he is not keeping any other bid from being granted.*

Note that a GVA would have allocated both goods to Green and made him pay 18*.*

9 Complex bidders

Our assumption that bidders are single-minded seems very restrictive, is there a way to extend our results to more complex players? Why not view a player as a collection of single-minded agents, or, equivalently, view the type of a player as a collection of bids? In such a setting, the game played becomes much richer in strategies and players may be better-off lying on some of their bids to obtain an advantage on others. Our discussion will, by necessity, be sketchy.

A further discussion of these issues can be left for a future paper since our result, concerning the greedy allocation's properties, is negative and based on a simple situation that can be embedded in any of the proposals above. A stronger result can be proved but space considerations do not allow us to present it here: no payment scheme makes the greedy allocation a truthful mechanism for complex bidders. Here, we shall present a concrete example.

Example 4 *The mechanism we consider is the greedy mechanism. Red is a single-minded bidder and his type is $\langle a, 12\rangle$, i.e. he bids* 12 *for a alone. Green is a complex bidder. His type t_G is described by: $t_G(\{a\}) = 10$, $t_G(\{b\}) = 10$ and $t_G(\{a, b\}) = 30$. Notice that Green exhibits complementarity: he values the set $\{a, b\}$ at more than the sum of his values for a and b. Whatever stance one takes about the way a set of single-minded bidders can, in general, represent a type, in this case, the set of three bids: $\langle\{a\}, 10\rangle$, $\langle\{b\}, 10\rangle$ and $\langle\{a, b\}, 30\rangle$ is a truthful representation of Green's type. Even if the rules of the auction allow the auctioneer to grant Green both his bid for a and his bid for b, Green cannot complain, in such a case, about his bid for the set $\{a, b\}$ being denied since he will, under any reasonable payment scheme, pay less for a and b separately than for his bid for the whole set. Suppose Green bids truthfully. The greedy mechanism grants Green's bid for the set $\{a, b\}$ and denies all other bids. Green pays* 24 *(in a GVA he would pay only* 12*), and therefore his utility is* 6*. To eliminate all doubts about the legitimacy of the* *payment scheme here, notice that Green's payment is determined by Red's bid, not by Green's other bids.*

But consider what would have happened if Green had under-bid and declared: $\langle\{a\}, 10\rangle$, $\langle\{b\}, 10\rangle$ *and* $\langle\{a, b\}, 23\rangle$. *The greedy mechanism now allocates* a *to Red (he pays* 11.5*) and* b *to Green. Green pays zero. His utility is* 10. *Green is better off lying. Notice that, by lying on his valuation for the set* $\{a, b\}$, *Green looses (*6*) on this bid: considered in isolation, this bid had no incentive to lie, but this lie favors the bid for* b *which happens to be Green's also.*

Example 4 above exhibits a situation in which a gang of single-minded players may be globally better off under-bidding and losing utility on one of its bids, in order to have another of the gang's bid granted and making up for the loss, and more. A similar situation can arise in which a gang may be better off over-bidding on a bid b_1 to ensure that it is granted, even at a loss, in order to keep another bidder from getting goods included in another bid of the gang.

10 Revenue considerations

We have described a feasible mechanism for combinatorial auctions that is truthful when bidders are single-minded. Should a seller use it for selling goods? It is very difficult to say anything general about the revenue generated by this mechanism. We shall compare the revenue generated by our mechanism to the revenue generated by a GVA. Since a GVA allocates the goods in an efficient way but our mechanism does not, one can fear that the revenue generated by our mechanism will be significantly smaller in all those cases in which the allocation is not efficient. This does not seem to be the case. There are cases in which our algorithm generates a higher revenue than a GVA and there are cases in which a GVA is preferable. The comparison does not seem to be tightly correlated to the relative efficiency of the allocations. Space considerations limit us to a single example.

Example 5 *Assume there are four goods,* a, b, c *and* d. *Green is interested in* $\{a, b\}$, *Red in* $\{c, d\}$ *and Black in* $\{a, c\}$. *All bids are for the same amount:* 1.

Let us first consider a GVA. A GVA allocates the efficient way: Green gets $\{a, b\}$ *and Red gets* $\{c, d\}$. *Green and Red pay nothing: if they had not* *participated only one bid could have been granted. The revenue generated by a GVA is zero.*

Because of the tie our greedy scheme may end in one of three possible situations, up to symmetry between Green and Red. First scenario: the order is Green, Red, Black. The allocation is efficient and nobody pays anything, as for a GVA. Second scenario: the order is Green, Black, Red. The allocation is efficient, but this times Green pays 1*, Red pays nothing. Third scenario: the order is Black, Green, Red. The allocation is* not *efficient: Black gets* $\{a, c\}$ *and* b *and* d *are unallocated. Black pays* 1*.*

In this case, our scheme generates, on average, 2/3*, whereas a GVA generates* 0*.*

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